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Consumption-Portfolio Choice with Preferences for Cash

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Non-Technical Summary

Following the classical work by Merton (1969, 1971), most papers that study consumption-portfolio decisions disregard money. However, as for instance pointed out by Obstfeld and Rogo (1996), p. 513, “many of the most intriguing and important questions in international finance involve money". Now, there are several ways to take money into account. One approach assumes that money enters the utility function. This idea of modeling preferences for liquidity is well established. Starting with Sidrauski (1967) and Brock (1974), a strand of macroeconomic literature uses “money in the utility function" to address economic problems involving monetary issues. Intuitively, this approach could be justified by the implicit assumption that the agent has utility from both consumption and leisure. Real money balance thus enters the utility function indirectly since agents save time in conducting their transactions when holding cash.
Despite its widespread use in the theory of monetary policy, a rigorous treatment of consumption-portfolio decisions with preferences for cash is missing. Our paper fills this gap and studies a canonical consumption-portfolio problem of a household with these preferences. We provide closed-form solutions in both the finite- and infinite-horizon case and also prove formal verification theorems showing that these solutions are indeed optimal.
We find that in the finite-horizon setting preferences for cash lead to time-dependent risky portfolio shares that are decreasing in age for reasonable calibrations. Such a pattern is qualitatively in line with rules of thumb that the risky share should decrease in age. It is also supported by recent empirical evidence showing that households who participate in the stock market have high and fairly constant risky shares during young ages, while investors reduce their risky share at a steady pace from about age 45 until they reach retirement (see Fagereng, Gottlieb, and Guiso (2017)).

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Consumption-Portfolio Choice with Preferences for Cash

ABSTRACT: This paper studies a consumption-portfolio problem where money enters the agent's utility function. We solve the corresponding Hamilton-Jacobi-Bellman equation and provide closed-form solutions for the optimal consumption and portfolio strategy both in an infinite- and finite-horizon setting. For the infinite-horizon problem, the optimal stock demand is one particular root of a polynomial. In the finite-horizon case, the optimal stock demand is given by the inverse of the solution to an ordinary differential equation that can be solved explicitly. We also prove verification results showing that the solution to the Bellman equation is indeed the value function of the problem. From an economic point of view, we find that in the finite-horizon case the optimal stock demand is typically decreasing in age, which is in line with rules of thumb given by financial advisers and also with recent empirical evidence. Our results are robust to introducing recursive utility.

KEYWORDS: consumption-portfolio choice, money in the utility function, stock demand, stochastic control, recursive utility

JEL-CLASSIFICATION: G11, C61

1 Introduction

Following the classical work by Merton (1969, 1971), most papers that study consumption-portfolio decisions disregard money. However, as for instance pointed out by Obstfeld and Rogoff (1996), p. 513, "many of the most intriguing and important questions in international finance involve money". Now, there are several ways to take money into account. One approach assumes that money enters the utility function. This idea of modeling preferences for liquidity is well established. Starting with Sidrauski (1967) and Brock (1974), a strand of macroeconomic literature uses "money in the utility function" to address economic problems involving monetary issues.[1] Recent papers in financial economics applying this approach involve Balvers and Huang (2009) and Gu and Huang (2013), among others. Intuitively, this approach could be justified by the implicit assumption that the agent has utility from both consumption and leisure. Real money balance thus enters the utility function indirectly since agents save time in conducting their transactions when holding cash.

Despite its widespread use in the theory of monetary policy, a rigorous treatment of consumption-portfolio decisions with preferences for cash is missing. Our paper fills this gap and studies a canonical consumption-portfolio problem of a household with these preferences. We provide closed-form solutions in both the finite- and infinite-horizon case and also prove formal verification theorems showing that these solutions are indeed optimal. For the infinite-horizon case, the optimal stock demand is characterized by a polynomial. More precisely, it is one particular root of this polynomial. For the finite-horizon case, we show that the optimal stock demand is given by the inverse of the solution to an ordinary differential equation. The solution to the differential equation can be calculated explicitly. Since individual optimality of household decisions is part of every general equilibrium analysis, our analysis also contributes to this field.

From an economic point of view, we find that in the finite-horizon setting preferences for cash lead to time-dependent risky portfolio shares that are decreasing in age for realistic calibrations. Such a pattern is qualitatively in line with rules of thumb that the risky share should decrease in age. It is also supported by recent empirical evidence showing that households who participate in the stock market have high and fairly constant risky shares during young ages, while investors reduce their risky share at a steady pace from about age 45 until they reach retirement (see Fagereng, Gottlieb, and Guiso (2017)). Finally, we also study the decision problem of an agent who has recursive utility and preferences for cash. In this setting, we provide evidence that our

[1]See, e.g., Obstfeld and Rogoff (1996), p. 530, for more details and references. Google Scholar reports more than 1,600 citations for Sidrauski (1967) and more than 600 for Brock (1974).

findings are robust to varying the elasticity of intertemporal substitution.

Our paper is related to Dixit and Goldman (1970) who also assume that money enters the agent's utility. However, they consider a discrete-time setup and analyze first-order conditions only. In contrast to our paper, they cannot solve the model further and do not provide closed-form solutions or a verification theorem. Other related papers that assume that money provides utility are Fama and Farber (1979) and LeRoy (1984a,b). Fama and Farber (1979) study an economy where the government provides money supply in a partial equilibrium model while LeRoy (1984a) and LeRoy (1984b) consider a similar general equilibrium setup. An alternative way to derive a demand for money is the cash-in-advance approach. In this approach, money is necessary to make transactions and does not directly affects the utility (see, e.g., Lucas (1982) and Svensson (1985)).

The remainder of the paper is structured as follows: Section 2 describes our framework. Section 3 provides and solves the Hamilton-Jacobi-Bellman equation of the agent's consumption-portfolio problem. It also studies some properties of the first-order conditions. Sections 4 and 5 analyze the infinite- and finite-horizon case in detail and derive the candidates for the optimal controls. Section 6 shows that these candidates are indeed optimal and proves the corresponding verification results. Section 7 studies the decision problem of an agent with recursive utility. Section 8 provides numerical examples both for time-additive and recursive utility. Section 9 concludes. All proofs can be found in the Appendix.

2 Framework

We consider a canonical portfolio problem where the agent can hold cash or invest in a stock index (short: stock). The dynamics are given by[2]

$$\begin{aligned} dM &= Mrdt, \\ dS &= S[\mu dt + \sigma dW]. \end{aligned}$$

The agent's wealth satisfies the dynamic budget constraint

$$dX = X[(r + \pi\eta)dt + \sigma\pi dW] - cdt \tag{2.1}$$

where $\eta = \mu - r$ and $\sigma > 0$ are the stock's excess return and volatility, π is the proportion invested in stock, and c denotes consumption. The agent maximizes expected utility from

[2]For notational convenience, we omit time indices whenever it is not necessary for clarity.

consumption and terminal wealth, but has also preferences for holding cash. This is because cash balances, by virtue of their liquidity, provide services (see, e.g., Obstfeld and Rogoff (1996) and the references therein). Therefore, the agent maximizes

$$\sup_{\pi,c} \int_0^T e^{-\delta t} \mathrm{E}[u(c_t, X_t(1-\pi_t))]\, dt + e^{-\delta T} \mathrm{E}[U(X_T, X_T(1-\pi_T))].$$

Notice that $X_t(1-\pi_t)$ equals the amount invested in cash at time t. We assume that the agent's utility functions are given by

$$u(c,\ell) = \frac{1}{1-\gamma}(c^\beta \ell^{1-\beta})^{1-\gamma}, \quad U(x,\ell) = \frac{1}{1-\gamma}(x^\alpha \ell^{1-\alpha})^{1-\gamma},$$

i.e. he has a power utility function where the trade-off between consumption c or wealth x and liquid funds ℓ is measured by a Cobb-Douglas function. The constants $\alpha \in (0,1]$ and $\beta \in (0,1)$ can be interpreted as weights.[3] For $\alpha = 1$, we obtain standard bequest $U(x) = x^{1-\gamma}/(1-\gamma)$ as a special case. Throughout, we make the standing assumption that the risk aversion coefficient satisfies

$$\gamma > 1. \tag{2.2}$$

This requirement is imposed since risk aversions below one are not reasonable from an economic point of view.[4] Besides, we assume that the agent's optimal stock demand in an ordinary Merton (1969, 1971) problem were positive, i.e.[5]

$$m \equiv \frac{\eta}{\gamma\sigma^2} > 0. \tag{2.3}$$

From an economic point of view, this assumption is not restrictive. The condition $m \geq 0$ is only violated if the expected excess return is negative, which is unrealistic and not in line with empirical estimates for major stock markets (e.g. US stock market). The case $m = 0$ is trivial in our setting since even without preferences for cash the agent invests all his funds in the money market account. Therefore, preferences for cash have no effect on his portfolio decisions and we disregard this case. Finally, for simplicity we assume that

$$r \geq -\frac{\delta}{\gamma - 1}, \tag{2.4}$$

[3]We exclude $\beta = 1$ since then the agent has no intermediate preferences for cash, which is the standard case if $\alpha = 1$. This avoids several case distinctions. The case $\beta = 1$ and $\alpha \in (0,1)$ is not particular interesting and thus disregarded.

[4]See, e.g., Munk (2013), pp. 175ff.

[5]Table 1 summarizes all relevant constants that are defined in this paper.

which is always satisfied for positive interest rates and can only be violated if the interest rate is sufficiently negative.[6]

3 Bellman Equation

The Hamilton-Jacobi-Bellman equation (short: Bellman equation) for our problem is given by

$$0 = \sup_{\pi,c} \left\{ G_t + xrG_x + x\pi\eta G_x + 0.5x^2\sigma^2\pi^2 G_{xx} - cG_x + u(c, x(1-\pi)) - \delta G \right\} \tag{3.5}$$

with terminal condition $G(T, x) = U(x, x(1-\pi(T)))$. The first-order condition for consumption becomes

$$c = \beta^{\frac{1}{\kappa}} (x(1-\pi))^{\frac{(\beta-1)(\gamma-1)}{\kappa}} G_x^{-\frac{1}{\kappa}}, \tag{3.6}$$

where $\kappa \equiv 1 + \beta(\gamma - 1) > 0$. The first-order condition for stock is

$$\pi = \left(\frac{\eta}{\sigma^2} - \frac{1-\beta}{\beta} \frac{1}{1-\pi} \frac{c}{x} \right) \frac{-G_x}{xG_{xx}}. \tag{3.7}$$

Representation (3.7) already indicates that the stock demand in our setting is reduced relatively to the ordinary model without preferences for cash since the second term in the bracket reduces the stock demand. This is because the agent is willing to give up part of the stock's excess return to hold cash. In fact, for a finite-horizon setting this generates a life-cycle pattern since the implicit price of holding cash typically becomes smaller over time. In the latter sections, we will see that this intuition can be formalized.

Furthermore, notice that these first-order conditions are still implicit and we will derive an explicit equation for the stock demand π in the latter sections, which then also allows us to calculate the optimal consumption via (3.6). By setting $\beta = 1$ we can recover the standard results of models without preferences for cash. Using the above relations, we can calculate some terms in the Bellman equation more explicitly

$$\begin{aligned} cG_x &= \beta^{\frac{1}{\kappa}} (x(1-\pi))^{\frac{(\beta-1)(\gamma-1)}{\kappa}} G_x^{\frac{\beta(\gamma-1)}{\kappa}}, \\ u(c, x(1-\pi)) &= \tfrac{1}{1-\gamma} \beta^{\frac{\beta(1-\gamma)}{\kappa}} (x(1-\pi))^{\frac{(\beta-1)(\gamma-1)}{\kappa}} G_x^{\frac{\beta(\gamma-1)}{\kappa}}. \end{aligned} \tag{3.8}$$

Using these results we can eliminate c in the Bellman equation:

$$0 = \sup_{\pi} \left\{ G_t + xrG_x + x\pi\eta G_x + 0.5x^2\sigma^2\pi^2 G_{xx} - \delta G + K(x(1-\pi))^{\frac{(\beta-1)(\gamma-1)}{\kappa}} G_x^{\frac{\beta(\gamma-1)}{\kappa}} \right\}$$

[6] Actually, condition (2.4) could be relaxed further. This condition is sufficient to ensure that $D > 0$ (see equation (B.44)). Instead we could also impose the condition that D is positive.

with

$$K \equiv \tfrac{1}{1-\gamma}\beta^{\frac{\beta(1-\gamma)}{\kappa}} - \beta^{\frac{1}{\kappa}} = \beta^{\frac{1}{\kappa}-1}\left(\tfrac{1}{1-\gamma} - \beta\right) = \beta^{\frac{1}{\kappa}-1}\tfrac{\kappa}{1-\gamma} < 0.$$

By (3.8), the first-order condition for the optimal stock demand (3.7) becomes

$$\eta + \sigma^2\pi\frac{xG_{xx}}{G_x} = \frac{1-\beta}{\beta}\beta^{\frac{1}{\kappa}}\Big((1-\pi)^{\gamma}x^{\gamma}G_x\Big)^{-\frac{1}{\kappa}}. \tag{3.9}$$

We now conjecture

$$G(t,x) = \frac{1}{1-\gamma}x^{1-\gamma}f(t)^k \tag{3.10}$$

with a constant k that will be chosen later on. The terminal condition on f is given by

$$f(T)^k = (1-\pi(T))^{(1-\alpha)(1-\gamma)}. \tag{3.11}$$

Then the first-order condition (3.6) for consumption becomes

$$\frac{c}{x} = \beta^{\frac{1}{\kappa}}\Big((1-\pi)^{(1-\beta)(\gamma-1)}f^k\Big)^{-\frac{1}{\kappa}} \tag{3.12}$$

and the first-order condition (3.9) for stock

$$\eta - \gamma\sigma^2\pi = \frac{1-\beta}{\beta}\beta^{\frac{1}{\kappa}}\Big((1-\pi)^{\gamma}f^k\Big)^{-\frac{1}{\kappa}}. \tag{3.13}$$

For the moment we make the *assumption* that the following conditions hold. A formal verification proof that these assumptions are actually satisfied can be found in Section 6.

(i) The consumption-portfolio problem has a smooth value function with $f \in C^1$, $f > 0$, and $k \in I\!R$.

(ii) The FOCs (3.12) and (3.13) determine the optimal consumption and stock demand.

(iii) If the horizon is infinite, then a suitable transversality condition holds.[7]

Under these assumptions, the optimal stock demand characterized by (3.13) is deterministic and independent of the agent's wealth. Notice that (3.13) is of the form

$$\eta - \gamma\sigma^2\pi(t) = h(t) \tag{3.14}$$

with a strictly positive deterministic function $h(t)$.

The following proposition establishes global concavity of our problem:

[7]See, e.g., Duffie (2001).

Proposition 3.1 (Concavity and Global Optimality)**.** *Assume that (i) holds. Then the function*

$$H(c,\pi) \equiv x\pi\eta G_x - cG_x + 0.5x^2\sigma^2\pi^2 G_{xx} + u(c, x(1-\pi))$$

is strictly concave, i.e. a solution to the FOCs (3.12) and (3.13) is a global maximum.

As already announced above when we discussed the representation of the optimal stock demand (3.7), we can now show that the optimal stock demand in our framework is always smaller than the demand in an ordinary setting without preferences for cash. More precisely, we obtain the following result.

Proposition 3.2 (Size of the Optimal Stock Demand)**.** *Under assumptions (i)-(iii) and conditions (2.2)-(2.4) the agent's stock demand is bounded from above as follows:*

$$\pi^*(t) < \min\{m, 1\}, \tag{3.15}$$

where m is the stock demand in a Merton problem defined in (2.3).

Notice that the result of the previous proposition is slightly sharper. It states that the demand must be smaller than one as well. This is a binding restriction if the stock investment is very beneficial leading to a stock demand above one in an ordinary Merton problem. In our setting, the agent always holds cash following from his preferences for cash and thus his demand is always below one.

Substituting the guess (3.10) into the Bellman equation yields

$$0 = \tfrac{k}{1-\gamma} f_t + \left(r - \tfrac{\delta}{1-\gamma} + \pi\eta - 0.5\gamma\sigma^2\pi^2\right) f + K(1-\pi)^{-\frac{(\beta-1)(1-\gamma)}{\kappa}} f^{1-\frac{k}{\kappa}}, \tag{3.16}$$

where we evaluate the Bellman equation at the optimal stock demand $\pi = \pi^*$. Hence, the separation (3.10) works, but the first-order condition (3.13) and the Bellman equation (3.16) constitute a coupled system of equations for f and π^*. Using (3.13) the Bellman equation can be rewritten as

$$0 = \tfrac{k}{1-\gamma} f_t + \left(r - \tfrac{\delta}{1-\gamma} + \pi\eta - 0.5\gamma\sigma^2\pi^2 + \widetilde{K}(\eta - \gamma\sigma^2\pi)(1-\pi)\right) f \tag{3.17}$$

with

$$\widetilde{K} \equiv K \frac{\beta}{1-\beta} \beta^{-\frac{1}{\kappa}} = \frac{\kappa}{(1-\gamma)(1-\beta)}.$$

By choosing k as

$$k = \kappa = 1 + \beta(\gamma - 1),$$

one can isolate f in (3.13):

$$f = \frac{\frac{1-\beta}{\beta}\beta^{\frac{1}{\kappa}}}{(\eta - \gamma\sigma^2\pi)(1-\pi)^{\frac{\gamma}{\kappa}}}. \tag{3.18}$$

Notice that the optimal stock demand thus directly determines the value function and vice versa. Furthermore, we can substitute into (3.12) and rewrite the consumption-wealth ratio

$$\frac{c}{x} = \frac{\beta}{1-\beta}(\eta - \gamma\sigma^2\pi)(1-\pi), \tag{3.19}$$

which is inversely related to the stock demand.

We can now derive an explicit characterization of the optimal stock demand via an ordinary differential equation. The key idea is to use the explicit representation (3.18) for f and then to substitute back into the Bellman equation:

Proposition 3.3 (ODE Characterization of the Optimal Stock Demand)**.**

$$\frac{d\pi}{dt} = \frac{\gamma-1}{k}\frac{P_2(\pi)(1-\pi)(m-\pi)}{(1-\pi)+\frac{\gamma}{\kappa}(m-\pi)} \tag{3.20}$$

where $P_2(\pi)$ is a second-order polynomial in π:

$$P_2(\pi) = r - \frac{\delta}{1-\gamma} + \pi\eta - 0.5\gamma\sigma^2\pi^2 + \widetilde{K}\gamma\sigma^2(m-\pi)(1-\pi). \tag{3.21}$$

The result of the previous proposition is crucial for studying the stock demand over the life-cycle. This is because the ODE (3.20) determines the derivative of the stock demand with respect to time t. If this derivative is negative, then the stock demand is decreasing over the life-cycle. In Proposition 3.2, we have already established that the stock demand is both smaller than one and the Merton result m, since our agent has preferences for cash. Consequently, the life-cycle behavior is determined by the polynomial P_2. This is formalized in the following proposition.

Proposition 3.4 (Slope of the Optimal Stock Demand)**.** *Suppose that assumptions (i)-(iii) hold. If conditions (2.2)-(2.4) are satisfied, then (3.15) holds and thus the denominator in (3.20) is always positive, i.e.*

$$(1-\pi) + \frac{\gamma}{\kappa}(m-\pi) > 0.$$

In particular, the sign of the polynomial P_2 determines the sign of the slope of π over time, i.e.

$$P_2(\pi) > 0 \iff \frac{d\pi}{dt} > 0 \quad \textit{and} \quad P_2(\pi) < 0 \iff \frac{d\pi}{dt} < 0.$$

To be able to discuss the economic implications of this result, we must characterize the solution of the ODE for the optimal stock demand. One of the important ingredients is its terminal condition. Since f must satisfy the terminal condition (3.11), the terminal condition for π can be calculated from (3.13) by solving for $\pi(T)$ in the following equation

$$\begin{aligned} \eta - \gamma\sigma^2\pi(T) &= \frac{1-\beta}{\beta}\beta^{\frac{1}{\kappa}}\Big((1-\pi(T))^{\gamma} f(T)^{k}\Big)^{-\frac{1}{\kappa}} \\ &= C\Big(1-\pi(T)\Big)^{-\frac{\kappa_\alpha}{\kappa}} \end{aligned} \tag{3.22}$$

with $\kappa_\alpha \equiv 1 + \alpha(\gamma - 1)$ and

$$C \equiv \frac{1-\beta}{\beta}\beta^{\frac{1}{\kappa}}. \tag{3.23}$$

There are two special cases where the terminal condition becomes explicit. The following proposition summarizes these cases.

Proposition 3.5 (Terminal Condition on Optimal Stock Demand)**.** *(i) If the liquidity preferences are the same for consumption and bequest, i.e. $\beta = \alpha$, we obtain*

$$\pi^*(T) = \frac{m+1}{2} - \sqrt{\frac{(m-1)^2}{4} + \frac{C}{\gamma\sigma^2}} < \min\{m, 1\}. \tag{3.24}$$

(ii) If $\eta = \gamma\sigma^2$, i.e. $m = 1$, then

$$\pi^*(T) = 1 - \left(\frac{C}{\gamma\sigma^2}\right)^{\frac{\kappa}{\kappa+\kappa_\alpha}}.$$

Now, we can formally solve the ODE (3.20) by applying separation of variables:

$$\frac{k}{\gamma-1}\int \frac{(1-\pi) + \frac{\gamma}{\kappa}(m-\pi)}{P_2(\pi)(1-\pi)(m-\pi)}d\pi = t + const, \tag{3.25}$$

where *const* can be determined by using terminal condition (3.22) on π. We first study the infinite-horizon case which is simpler since the optimal stock demand is constant. The solution of this case allows us to solve the more involved finite-horizon problem as well.

4 Infinite Horizon

For an infinite horizon, the ODE (3.20) simplifies to an algebraic equation for the optimal stock demand:

$$0 = \frac{\gamma-1}{k}\frac{P_2(\pi)(1-\pi)(m-\pi)}{(1-\pi) + \frac{\gamma}{\kappa}(m-\pi)}. \tag{4.26}$$

Notice that the optimal stock demand and in turn the function f is now simply a constant. Under assumptions (i)-(iii) the optimal demand is a root of the polynomial. In general, there are two possible solutions and we have to identify the correct one. Therefore, we study the polynomial in more detail which can be rewritten as follows

$$P_2(\pi) = \widehat{K}(\pi^2 + p\pi + q)$$

with

$$\widehat{K} \equiv \gamma\sigma^2(\widetilde{K} - 0.5) < 0, \quad p \equiv \frac{\eta - \widetilde{K}(\eta + \gamma\sigma^2)}{\widehat{K}}, \quad q \equiv \frac{r - \frac{\delta}{1-\gamma} + \widetilde{K}\eta}{\widehat{K}}. \tag{4.27}$$

The two candidate solutions are

$$\pi_{01/02} = -0.5p \pm \sqrt{D} \quad \text{with} \quad D \equiv p^2/4 - q.$$

The upper bound established in Proposition 3.2 is again crucial since only one solution is below this bound and is thus feasible. The following proposition formalizes our arguments:

Proposition 4.1 (Optimal Stock Demand for an Infinite Horizon)**.** *Under assumptions (i)-(iii) and conditions (2.2)-(2.4) the optimal stock demand for an infinite horizon is given by*

$$\pi^\infty \equiv \pi_{02} = -0.5p - \sqrt{D} < \min\{m, 1\}.$$

Given our previous results the polynomial of degree four in the numerator on the right-hand side of (3.20) has four real-valued roots:

$$\pi_{01} > \min\{m, \tfrac{1+m}{2}\}, \quad \pi^\infty < \min\{m, 1\}, \quad \pi_{03} = 1, \quad \pi_{04} = m, \tag{4.28}$$

where the last two become a double root for $m = 1$. It can thus be written as

$$P_2(\pi)(1-\pi)(m-\pi) = \widehat{K}(\pi_{01} - \pi)(\pi^\infty - \pi)(1-\pi)(m-\pi).$$

For $m = 1$ the representations in (4.27) become

$$p = -2, \qquad q = 1 + \frac{r - \frac{\delta}{1-\gamma} + 0.5\gamma\sigma^2}{\widehat{K}} \qquad \Longrightarrow \qquad D = (\widetilde{r} + 1)\frac{\gamma - \kappa}{\gamma + \kappa} > 0 \tag{4.29}$$

with $\widetilde{r} \equiv 2(r + \frac{\delta}{\gamma-1})/(\gamma\sigma^2) = 2(r + \frac{\delta}{\gamma-1})/\eta$. Therefore,

$$\pi_{01} = 1 + \sqrt{D}, \qquad \pi_{02} = 1 - \sqrt{D}.$$

Therefore, the representation of the optimal stock demand simplifies significantly:

Corollary 4.2 (Optimal Stock Demand for an Infinite Horizon and $m = 1$)**.** *Under assumptions (i)-(iii) and conditions (2.2)-(2.4) the optimal stock demand for an infinite horizon and $m = 1$ is given by*

$$\pi^{\infty} = \pi_{02} = 1 - \sqrt{D}.$$

Since the optimal demand in an ordinary Merton problem is one, we now get the clean result that the adjustment for liquidity preferences is given by $\sqrt{D}$. If D is larger, then the stock demand is smaller. Notice that D increases with δ, r, or β, i.e. the stock demand is inversely related to these parameters. This is intuitive since δ models time preferences and the higher δ the less the agent cares about the future and thus consumes more. Since preferences for cash link the stock demand to consumption, the agent invests less in stocks. Besides, higher interest rates make it more attractive to hold cash and thus the stock demand decreases. Finally, the cash demand increases in the strength of the preferences for cash captured by β and thus the stock demand is smaller for higher values of β. The effect of γ is ambiguous for time-additive utility since γ has a dual role and captures the agent's risk aversion, but also his elasticity of intertemporal substitution (EIS). We are able to study this issue in more detail in a latter section where we consider an agent with recursive utility. This allows us to disentangle risk aversion from EIS.

5 Finite Horizon

We first study the finite-horizon case for $m \neq 1$. The case $m = 1$ is analyzed below. Furthermore, notice that if $\pi^*(T) = \pi^{\infty}$, then the optimal stock demand is constant and equal to $\pi^*(t) = \pi^{\infty}$. This follows directly from (3.20). We thus impose

$$\pi^*(T) \neq \pi^{\infty} \tag{5.30}$$

Throughout this section, we also assume that assumptions (i)-(iii) and conditions (2.2)-(2.4) are satisfied.

5.1 The Case $m \neq 1$

Given our previous results we can rewrite (3.25):

$$\mathcal{G}(\pi) \equiv \int \frac{(1-\pi) + \frac{\gamma}{\kappa}(m-\pi)}{(\pi_{01}-\pi)(\pi^{\infty}-\pi)(1-\pi)(m-\pi)} d\pi = -\overline{K}t + const \tag{5.31}$$

with $\overline{K} \equiv -\widehat{K}\frac{\gamma-1}{k} > 0$. From a stochastic-control point of view, the function $\mathcal{G}$ is important in the sequel because the optimal stock demand is essentially the inverse of $\mathcal{G}$, which will be established below (see Proposition 5.2). Since the value function is determined by π via (3.18), it is crucial that $\mathcal{G}$ is well-defined, sufficiently smooth, and invertible. This is established in the following proposition.

Proposition 5.1 ($\mathcal{G}$ Well-Defined and Invertible)**.** *(i) The mapping $\mathcal{G}$ is a well-defined real-valued function on $(-\infty, \min\{m,1\})\backslash\{\pi^\infty\}$ with the representation*

$$\mathcal{G}(\pi) \equiv -\alpha_1 \ln(\pi_{01} - \pi) - \alpha_2 \ln|\pi^\infty - \pi| - \alpha_3 \ln(1-\pi) + \alpha_4 \ln(m - \pi), \tag{5.32}$$

where

$$\alpha_1 \equiv \frac{1 + \frac{\gamma}{k}m - \pi_{01}(1+\frac{\gamma}{k})}{(\pi_{01} - \pi^\infty)(1-\pi_{01})(\pi_{01} - m)}, \quad \alpha_2 \equiv \frac{1 + \frac{\gamma}{k}m - \pi^\infty(1+\frac{\gamma}{k})}{(\pi_{01} - \pi^\infty)(1-\pi^\infty)(m - \pi^\infty)},$$
$$\alpha_3 \equiv \frac{\frac{\gamma}{k}}{(1-\pi_{01})(1-\pi^\infty)}, \quad \alpha_4 \equiv \frac{1}{(\pi_{01} - m)(m - \pi^\infty)}.$$

(ii) For all $m > 0$, we have $\alpha_2 > 0$. If $m \in (0,1]$, then $\alpha_4 > 0$. If $m > 1$, then $\alpha_3 < 0$.

(iii) $\mathcal{G}(\pi) \to \infty$ if $\pi \searrow \pi^\infty$ or $\pi \nearrow \pi^\infty$.

(iv) If $m \in (0,1]$, then $\mathcal{G}(\pi) \to -\infty$ for $\pi \nearrow m$. If $m > 1$, then $\mathcal{G}(\pi) \to -\infty$ for $\pi \nearrow 1$.

(v) $\mathcal{G}'(\pi) > 0$ for $\pi < \pi^\infty$ and $\mathcal{G}'(\pi) < 0$ for $\pi \in (\pi^\infty, m]$. In particular, $\mathcal{G}$ has a well-defined inverse $\mathcal{G}^{-1}$ for $\pi < \pi^\infty$ and for $\pi \in (\pi^\infty, m]$.

The constant in (5.31) is given by $\pi^*(T)$, which can be determined by (3.22). Therefore, we obtain

$$const = \mathcal{G}(\pi^*(T)) + \overline{K}T \quad \Longrightarrow \quad \mathcal{G}(\pi^*(t)) = \overline{K}(T-t) + \mathcal{G}(\pi^*(T)),$$

where $\mathcal{G}(\pi^*(T))$ can be calculated by substituting $\pi^*(T)$ into (5.32). The optimal demand π^* must be smooth since it is directly related to f, which must be a C^1-function. Therefore, the properties of $\mathcal{G}$ imply the following result:

Proposition 5.2 (Size and Slope of the Optimal Stock Demand)**.** *(i) If $\pi^*(T) < \pi^\infty$, then $\pi^*(t) < \pi^\infty$ and $d\pi^*(t)/dt < 0$. (ii) If $\pi^*(T) \in (\pi^\infty, \min\{m,1\})$, then $\pi^*(t) \in (\pi^\infty, \min\{m,1\})$ and $d\pi^*(t)/dt > 0$. In any case, if (5.30) holds, then the optimal stock demand is given by*

$$\pi^*(t) = \mathcal{G}^{-1}(\overline{K}(T-t) + \mathcal{G}(\pi^*(T))). \tag{5.33}$$

In particular, π^* is a smooth C^1-function, which by (3.18) is also true for f.

Remark. Figure 1 that is discussed in Section 8 shows an example of the function $\mathcal{G}$ for the benchmark calibration reported in Table 2.

The relation between the terminal condition $\pi^*(T)$ and the optimal demand π^∞ in an infinite horizon setting is crucial for the question of whether the stock demand is increasing or decreasing over the life-cycle. Formally, the reason is that the form of the ODE (3.20) induces a monotonic behavior of the demand, since the slope is either decreasing or increasing, but cannot change its sign.

In all our numerical examples presented in Section 8, the first case with $\pi^*(T) < \pi^\infty$ obtains, whereas the second case can only occur if the parameters take extreme and unrealistic values (e.g. for an unrealistically high time-preference rate δ). From an economic point of view, the first case generates an interesting stock demand that is falling over the life-cycle. So adding preferences for liquidity to a standard Merton framework can generate a decreasing stock demand.

To understand the economic intuition behind the condition $\pi^*(T) < \pi^\infty$, we consider the case $m = 1$ where, as we will see in the following section, the same condition applies. For simplicity, we also assume $\alpha = \beta$, but choosing $\alpha = 1$ does not change the interpretation given below. By Propositions 3.5 and 4.2, the condition is then equivalent to

$$\frac{C}{\gamma\sigma^2} > D \qquad \Longleftrightarrow \qquad \beta^{\frac{1}{\kappa}-1} > \left(2(r + \tfrac{\delta}{\gamma-1}) + \eta\right) \underbrace{\frac{\gamma - 1}{\gamma + \kappa}}_{<1}$$

where we use (3.23), (4.29), and $\eta = \gamma\sigma^2$ since $m = 1$. For a given excess return η, the last inequality imposes a condition on β, r, and δ. If the agent has moderate preferences for cash,[8] i.e. $\beta \in (0.95, 1)$, then the left-hand side is close to one. Therefore, the inequality can only be violated if the interest rate or the time-preference rate is extremely high. High values of r incentivize the agent to invest a lot in the money market account already early in life. High values of δ amplify the consumption motive. Both effects crowd out the stock demand early in life and reverse its downward sloping pattern. In the general setting where $m \neq 1$, these findings are still valid as long as β remains moderate and the other parameters do not take unrealistically high or low values.

[8] In our numerical examples, we use values of β in this range leading to stock demands in a realistic range.

5.2 The Case $m = 1$

For the case $m = 1$ we have $\eta = \gamma\sigma^2$ and thus equation (5.31) simplifies into

$$-\frac{2(1-\beta)}{\gamma\sigma^2}\int \frac{1}{(\pi_{01}-\pi)(\pi^\infty-\pi)(1-\pi)}d\pi = t + const \tag{5.34}$$

because $-(1+\frac{\gamma}{\kappa})/\overline{K} = -2(1-\beta)/(\gamma\sigma^2)$. Since

$$\int \frac{1}{(\pi_{01}-\pi)(\pi^\infty-\pi)(1-\pi)}d\pi = \frac{2\ln(1-\pi) - \ln|\pi^\infty - \pi| - \ln(\pi_{01}-\pi)}{2D}$$

we can rewrite (5.34) as follows

$$\mathcal{H}\big(\pi^*(t)\big) = \mathcal{H}\big(\pi^*(T)\big)e^{\frac{D\gamma\sigma^2}{1-\beta}(T-t)}, \tag{5.35}$$

where[9]

$$\mathcal{H}(\pi) \equiv \frac{(1-\pi)^2}{|\pi^\infty-\pi|(\pi_{01}-\pi)} = \frac{(1-\pi)^2}{|1-\sqrt{D}-\pi|(1+\sqrt{D}-\pi)}, \quad \pi \in (-\infty, \pi^\infty) \cup (\pi^\infty, 1),$$

and D is given in (4.29). The constant $\mathcal{H}\big(\pi^*(T)\big)$ reads

$$\mathcal{H}\big(\pi^*(T)\big) = \frac{\left(\frac{C}{\gamma\sigma^2}\right)^{\frac{2\kappa}{\kappa+\kappa_\alpha}}}{\left|D - \left(\frac{C}{\gamma\sigma^2}\right)^{\frac{2\kappa}{\kappa+\kappa_\alpha}}\right|}, \tag{5.36}$$

where C is given by (3.23). In the special case $\alpha = \beta$, this simplifies further

$$\mathcal{H}\big(\pi^*(T)\big) = \frac{C}{|D\gamma\sigma^2 - C|}.$$

Again it is not a priori clear whether $\pi^*(T) > \pi^\infty$ or $\pi^*(T) < \pi^\infty$. As we show in the Appendix, the function $\mathcal{H}$ is invertible. Consequently, we get a fully explicit representation of the optimal stock demand:

Proposition 5.3 (Optimal Stock Demand for $m = 1$)**.** *(i) If $\pi^*(T) \in (-\infty, \pi^\infty)$, the optimal stock demand is explicitly given by*

$$\pi^*(t) = 1 - \sqrt{\frac{D}{1 - e^{-\frac{D\gamma\sigma^2}{1-\beta}(T-t)}/\mathcal{H}\big(\pi^*(T)\big)}}.$$

[9] The case $\pi = \pi^\infty$ is not relevant due to assumption (5.30).

(ii) If $\pi^*(T) \in (\pi^\infty, 1)$, the optimal stock demand is explicitly given by

$$\pi^*(t) = 1 - \sqrt{\frac{D}{1 + e^{-\frac{D\gamma\sigma^2}{1-\beta}(T-t)}/\mathcal{H}\big(\pi^*(T)\big)}}.$$

In both cases, the constant $\mathcal{H}(\pi^*(T))$ is given by (5.36).

Remarks. a) If the horizon goes to infinity, i.e. $T \to \infty$, we recover the result of Corollary 4.2.

b) By construction of the inverse, the right-hand side in both cases becomes $\pi^*(T)$ if $t = T$.

c) By differentiating $\pi^*(t)$ w.r.t. time t, one can check that for (i) the stock demand is increasing over time, whereas it is decreasing for (ii), which is in line with our previous results.

6 Verification

We now prove that the assumptions (i)-(iii) imposed in Section 3 are satisfied for our problem, i.e. the candidate G is indeed the value function and (π^*, c^*) is the optimal strategy. Notice that we will sometimes use the notation $X^{\pi,c}$ to emphasize that the wealth dynamics depend on the controls π and c. First, we need to define the set of admissible strategies.

Definition 6.1 (Admissible Strategy)**.** *A strategy (π, c) is said to be admissible if the following conditions are satisfied:*

(i) The processes π and c are progressively measurable w.r.t. the filtration generated by the Brownian motion W,

(ii) for all initial conditions $(t_0, x_0) \in [0, T] \times (0, \infty)$ the wealth equation (2.1) with $X^{\pi,c}(t_0) = x_0$ has a pathwise unique solution $\{X_t^{\pi,c}\}_{t\in[t_0,T]}$,

(iii) $X^{\pi,c} \geq 0$,

(iv) $c \geq 0$ and $\pi \leq 1$

(v) $c \leq \bar{c}X^{\pi,c}$ and $\pi \geq \underline{\pi}$ for constants $\bar{c}$ and $\underline{\pi}$ that can be different for different strategies.

We denote the set of admissible strategies by $\mathcal{A}$.

Remarks. a) Notice that (i)-(iii) are for instance satisfied if c and π are smooth deterministic functions.

b) If the conditions in (iv) do not hold, then the utility functional is not well-defined.

c) The conditions in (v) are imposed to simplify matters. They could be relaxed by imposing suitable integrability conditions on c and π. From an economic point of view, it seems however reasonable that the stock demand is bounded from below and that the consumption-wealth ratio is bounded from above.

First, we summarize our previous results concerning the candidates for the value function and the optimal controls.

Theorem 6.2 (Candidates for Value Function and Controls)**.** *Assume that conditions (2.2)-(2.4) hold. Then we obtain the following: (i) The candidate (5.33) for the optimal stock demand* π^* *is a well-defined real-valued* C^1*-function defined on* $[0,T]$ *and mapping into* $(-\infty, \pi^\infty)$ *if* $\pi^*(T) < \pi^\infty$ *and mapping into* (π^∞, m) *if* $\pi^*(T) > \pi^\infty$*.*

(ii) The candidates f *and* c^* *are well-defined real-valued* C^1*-functions given by (3.18) and (3.19).*

(iii) The pair (π^*, c^*) *is admissible in the sense of Definition 6.1.*

Now, we can verify that the candidates for the optimal stock demand and consumption are indeed optimal.

Theorem 6.3 (Verification for a Finite Horizon)**.** *The function* G *given by (3.10) is the value function of the problem and* π^* *and* c^* *are the optimal stock demand and consumption.*

If the horizon is infinite, then the value function and the optimal strategy become time-independent. In particular, G satisfies the Bellman equation (3.5) where the time derivative G_t is set to zero. The candidate for the optimal stock demand is the constant π^∞ and satisfies the algebraic equation (4.26) that we have analyzed in detail in Section 4. In turn, the candidates for f and c/x that are still given by (3.18) and (3.19) are also constant. To ensure that our verification result provided in Theorem 6.3 carries over to the infinite-horizon case, we have to prove a so-called transversality result.[10] If this is satisfied, then loosely speaking beqeust becomes negligible if the horizon gets larger. Formally, we obtain the following result:

Theorem 6.4 (Verification for an Infinite Horizon)**.** *Assume that the candidate for the optimal stock demand satisfies* $\pi^\infty > 0$*. Then* π^∞ *is the optimal stock demand and the value function is given by*

$$G(x) = \frac{1}{1-\gamma} x^{1-\gamma} f^k$$

where f *is a constant that is determined by* π^∞ *via (3.18). The optimal consumption-wealth ratio* c^∞/x *is also constant and given by*

$$\frac{c^\infty}{x} = \frac{\beta}{1-\beta}(\eta - \gamma\sigma^2\pi^\infty)(1-\pi^\infty). \tag{6.37}$$

[10]See, e.g., Duffie (2001), p. 213.

Remark. The condition $\pi^\infty > 0$ excludes pathological cases where the candidate for the optimal stock demand is negative. This can only occur if β becomes very small.

7 Recursive Utility

We now consider an agent with recursive preferences that are referred to as stochastic differential utility (SDU) in continuous time. Duffie and Epstein (1992) and Kraft and Seifried (2014) show that the aggregator inducing a continuous-time version of Epstein-Zin preferences reads

$$\mathcal{F}(\mathcal{C}, v) = \delta v \Big(\mathcal{C}^{1-\phi} ((1-\gamma)v)^{-\frac{1}{\theta}} - \theta \Big), \tag{7.38}$$

where $\theta = \frac{1-\gamma}{1-\phi}$ and $\delta, \gamma, \phi > 0$. The aggregator involves three parameters, the time-preference rate δ, the risk aversion γ, and the elasticity of intertemporal substitution (EIS) given by $1/\phi$. Therefore, Epstein-Zin preferences allow us to disentangle risk aversion from EIS. Notice that for time-additive utility this is not possible since $\phi = \gamma$. Nevertheless, (7.38) thus contains the preferences introduced in Section 2 as a special case. The utility of a given consumption-portfolio strategy (c, π) is recursively defined by

$$G^{c,\pi}(t,x) \equiv \mathcal{V}_t^{c,\pi} = \mathrm{E}_t \Big[\int_t^T \mathcal{F}(\mathcal{C}_s^{c,\pi}, \mathcal{V}_s^{c,\pi})\, ds \Big],$$

where $\mathcal{C}_t^{c,\pi} \equiv c_t^\beta \big(X_t(1-\pi_t) \big)^{1-\beta}$. In the following, we drop the indices c, π for notational convenience. Duffie and Epstein (1992) show that the analogue of the Bellman equation (3.5) is given by

$$0 = \sup_{\pi,c} \big\{ G_t + xrG_x + x\pi\eta G_x + 0.5x^2\sigma^2\pi^2 G_{xx} - cG_x + \mathcal{F}(\mathcal{C}, G) \big\} \tag{7.39}$$

with terminal condition $G(T,x) = U(x, x(1-\pi(T)))$. Notice that the aggregator of time-additive utility is given by

$$u(\mathcal{C}) - \delta v,$$

i.e. the time-additive aggregator is replaced by the Epstein-Zin aggregator. The first-order condition for consumption becomes

$$c = \beta^{\frac{1}{\kappa_\phi}} (x(1-\pi))^{\frac{(\beta-1)(\phi-1)}{\kappa_\phi}} G_x^{-\frac{1}{\kappa_\phi}} ((1-\gamma)G)^{\frac{\theta-1}{\theta\kappa_\phi}}, \tag{7.40}$$

where $\kappa_\phi \equiv 1 + \beta(\phi - 1) > 0$. The first-order condition for stock is the same as before, i.e. it is identical to (3.7). Notice that, although we get the sane first-order condition for stock, this does not mean that the stock demand has to be the same, since the function G is different from the one in the previous sections. Nevertheless, this result already indicates that the stock demand might not fundamentally change for recursive utility. We will analyze this issue in Section 8 in more detail. Using (7.40) the first-order condition for stock can be rewritten as

$$\eta + \sigma^2 \pi \frac{x G_{xx}}{G_x} = \frac{1-\beta}{\beta} \beta^{\frac{1}{\kappa_\phi}} \left((1-\pi)^\phi x^\phi G_x ((1-\gamma)G)^{\frac{1-\theta}{\theta}} \right)^{-\frac{1}{\kappa_\phi}}.$$

As before we use the conjecture (3.10) with terminal condition (3.11). Then the first-order condition for consumption becomes

$$\frac{c}{x} = \beta^{\frac{1}{\kappa_\phi}} \left((1-\pi)^{(1-\beta)(\phi-1)} f^{\frac{k}{\theta}} \right)^{-\frac{1}{\kappa_\phi}} \tag{7.41}$$

and the first-order condition for stock

$$\eta - \gamma \sigma^2 \pi = \frac{1-\beta}{\beta} \beta^{\frac{1}{\kappa_\phi}} \left((1-\pi)^\phi f^{\frac{k}{\theta}} \right)^{-\frac{1}{\kappa_\phi}}. \tag{7.42}$$

These first-order conditions are similar to the ones for time-additive utility, but γ is replaced by ϕ whenever it plays the role of 1/EIS. This is only not happening if γ is really capturing risk aversion (e.g. in front of the variance σ^2). In Appendix E we show that the ODE for the optimal stock demand now becomes:

Proposition 7.1 (ODE Characterization of Optimal Stock Demand for SDU)**.** *The optimal stock demand satisfies*

$$\frac{d\pi}{dt} = \frac{\phi - 1}{\kappa_\phi} \frac{P_2(\pi)(1-\pi)(m-\pi)}{(1-\pi) + \frac{\phi}{\kappa_\phi}(m-\pi)}, \tag{7.43}$$

where the second-order polynomial P_2 is

$$P_2(\pi) = r - \frac{\delta}{1-\phi} + \pi\eta - 0.5\gamma\sigma^2\pi^2 + \widetilde{K}_\phi \gamma \sigma^2 (m-\pi)(1-\pi).$$

with $\widetilde{K}_\phi \equiv \frac{\kappa_\phi}{(1-\phi)(1-\beta)}$. The terminal condition is given by (E.57).

Remark. One can check that all arguments in Sections 4 and 5 go through one by one for $\phi > 1$ where at obvious places γ must be replaced by ϕ. For $\phi < 1$ some arguments have to be modified. Furthermore, verification theorems can be proved extending the ideas in Duffie and Epstein (1992) or Kraft, Seiferling, and Seifried (2017) to a setting with preferences for cash. This is beyond the scope of this paper.

8 Numerical Examples

To illustrate the effect of preferences for cash, we now consider the stock demand for a baseline calibration and also perform some comparative statics. Our calibration is summarized in Table 2. The values of the real interest rate r, the equity premium η, and the stock volatility σ are in line with US data starting in 1960. As stock market data, we use returns on the CRSP value-weighted market portfolio inclusive of the NYSE, AMEX, and NASDAQ markets (cum dividend). The risk-free asset is estimated from the Treasury bill yield provided by the Risk Free File on CRSP Bond tape. To obtain real values, all time-series are deflated using the consumer price index (CPI) taken from the website of the Bureau of Labor Statistics. We slightly reduce the equity premium to 5% to account for survivorship bias (Brown, Goetzmann, and Ross (1995)) as well as the decline in discount rates and the implied unexpected capital gains over the sample period (Fama and French (2002)). An estimate of 5% is also in line with recent empirical evidence by Avdis and Wachter (2017) suggesting that the sample average excess return is overstating the equity premium. The choices of the risk aversion $\gamma = 3$ and time-preference rate $\delta = 0.03$ are in line with the literature on life-cycle portfolio choice (see, e.g., Munk (2013) and the references therein), but we also vary the risk aversion γ and the consumption weight β. We assume that $\alpha = 1$, i.e. bequest does not involve preferences for cash. However, choosing for instance $\alpha = \beta = 0.99$ hardly changes our results. The time horizon is 50 years (e.g. age 25 until age 75).

We first illustrate how the function $\mathcal{G}$ given by (5.32) looks like. Recall that it plays an important role for the optimal stock demand. Figure 1 depicts $\mathcal{G}$ for our benchmark calibration. As we have seen in Section 5, the shape of of $\mathcal{G}$ is crucial for the size and slope of the optimal stock demand over time. In particular, the location of the pole of $\mathcal{G}$, which is the solution π^∞ to the infinite-horizon case, relative to the terminal condition $\pi^*(T)$ determines whether the stock demand is increasing or decreasing. As explained at the end of Section 5.1, this is determined by the values of δ, r, and β. More precisely, the stock demand is decreasing over the life-cycle if all parameters have moderate values of realistic size. As we are going to see now, this also leads to stock demands in realistic ranges.

In our benchmark case, we get $\pi^*(T) = 0.3867$, which is smaller than the optimal demand π^∞ of the infinite-horizon problem. Therefore, by Proposition 5.2, the left branch of the function $\mathcal{G}$ must be inverted to obtain the optimal stock demand, which is then decreasing over time. If $\pi^*(T)$ were larger than π^∞, then the right branch of $\mathcal{G}$ would be relevant and the stock demand would be increasing. Notice that $\mathcal{G}$ is increasing before the pole π^∞ and decreasing afterwards.

This is the reason why the optimal stock demand is decreasing or increasing depending on whether $\pi^\infty > \pi^*(T)$ or $\pi^\infty < \pi^*(T)$.

For our benchmark calibration, Figure 2 depicts the optimal demand m of an ordinary Merton problem and the optimal demands π^∞ and $\pi^*(t)$ of the infinite-horizon as well as the finite-horizon problem. As expected for the finite-horizon problem, we obtain a stock demand that is decreasing over time. Since the initial time horizon is large, it starts close to the solution of the infinite-horizon case, which is about 0.57, and decreases at an increasing rate to the terminal condition $\pi^*(T) = 0.3867$, i.e. the optimal stock demand is about 20% lower at the investment horizon. From an economic point of view, the time dependence comes from the fact that the relative price of liquidity, which is implicitly determined at the optimum, changes over time.

Figure 3 depicts the corresponding results if the stock demand in the Merton problem is 100%. This is achieved by increasing the equity premium to 0.0867, which is of course on high side. The case $m = 1$ is particularly interesting since any derivation from 100% can be interpreted as resulting from the preferences for cash. The effect is about 7% for the infinite-horizon case, but is much more pronounced for the finite-horizon case. In fact, the optimal stock demand decreases from about 92% to 65% over time leading to a liquidity effect between 7% and 35%.

Figure 4 depicts a calibration where the Merton demand is 110%. It can be seen that already in the infinite-horizon case preferences for cash bring down the optimal demand to less than 100%. This is in line with Proposition 4.1. Otherwise, the demand would not be admissible. For the finite-horizon problem, the optimal stock demand decreases from 95% to 70%.

Figure 5 shows comparative statics if we increase the weight of the cash preferences by decreasing the weight of consumption. Of course, the general effect is negative, i.e. more weight on cash preferences leads to less stock demand. However, it can be seen that the effect is initially pretty small, but at the end the terminal conditions are more sensitive. In particular, the optimal stock demands at the investment horizon vary between 38.7% and 6.8%. Finally, Figure 6 depicts comparative statics if we vary the risk aversion coefficient. Now, all stock demands decrease in risk aversion, which is reasonable.

Figures 7 and 8 assume that the agent has recursive utility as introduced in Section 7. Therefore, we can vary the elasticity of intertemporal substitution (EIS) independently of the risk aversion. Figure 7 shows that the stock demands are almost identical for different values of ϕ, i.e. the effect of the EIS on our results is negligible. This result is not unreasonable: Without preferences for cash the stock demand does not depend on the EIS at all. In our setting the stock demand is directly related to consumption because it enters the utility function of the agent. Therefore, there could potentially be an effect. However, it can be shown that in a Merton setting the consumption-wealth ratio for an agent with recursive utility, but without preferences for cash does not change significantly if the EIS is varied. Consequently, the effect on the stock demand is supposed to be moderate as well, which we indeed document in Figure 7. On the other hand, Figure 8 confirms that varying the risk aversion, but keeping the EIS fixed has a significant effect. In fact, the results of Figures 6 and 8 are almost identical since the EIS plays a negligible role. To summarize, these findings indicate that our previous results are robust to introducing recursive utility.

9 Conclusion

This paper studies a canonical consumption-portfolio problem with "money in the utility function". We provide the solution to the finite-horizon setting and analyze the relation to the problem with an infinite horizon. We show that for a finite horizon the stock demand can be increasing or decreasing over the life cycle where the latter result typically arises for realistic calibrations. Therefore, adding preferences for cash to an ordinary Merton problem can generate life-cycle stock demands that are in line with rules of thumb for stock investing and recent empirical evidence. This could potentially be a very useful result for modeling life-cycle consumption-portfolio decisions. We also consider an agent with recursive utility and preferences for cash. In this setting, we show that our findings are robust to varying the elasticity of intertemporal substitution.

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A Proofs of Section 3

Proof of Proposition 3.1. The sufficient conditions for a maximum are

$$H_{cc} < 0, \quad H_{\pi\pi} < 0, \quad \det \begin{bmatrix} H_{cc} & H_{c\pi} \\ H_{\pi c} & H_{\pi\pi} \end{bmatrix} > 0$$

with

$$\begin{aligned} H_{cc} &= u_{cc} = \beta(\beta(1-\gamma)-1)c^{\beta(1-\gamma)-2}(x(1-\pi))^{(1-\beta)(1-\gamma)} < 0, \\ H_{\pi\pi} &= -\gamma x^{1-\gamma}\sigma^2 f^k + (1-\beta)[(1-\beta)(1-\gamma)-1]c^{\beta(1-\gamma)}(x(1-\pi))^{(1-\beta)(1-\gamma)-2}x^2 < 0, \\ H_{c\pi} &= H_{\pi c} = \beta(1-\beta)(1-\gamma)c^{\beta(1-\gamma)-1}(x(1-\pi))^{(1-\beta)(1-\gamma)-1}(-x). \end{aligned}$$

Hence,

$$\begin{aligned} &\det \begin{bmatrix} H_{cc} & H_{c\pi} \\ H_{\pi c} & H_{\pi\pi} \end{bmatrix} \\ &= -\gamma x^{1-\gamma}\sigma^2 f^k \beta[\beta(1-\gamma)-1]c^{\beta(1-\gamma)-2}(x(1-\pi))^{(1-\beta)(1-\gamma)}x^2 \\ &\quad +\beta(1-\beta)[(1-\beta)(1-\gamma)-1][\beta(1-\gamma)-1]c^{2\beta(1-\gamma)-2}(x(1-\pi))^{2(1-\beta)(1-\gamma)-2}x^2 > 0. \end{aligned}$$

□

Proof of Proposition 3.2. For $m \leq 1$ we obtain $\pi^*(t) < m$ since (3.14) holds. For $m > 1$ the FOC (3.13) dictates $\pi^*(t) < 1$. □

Proof of Proposition 3.3. Taking derivatives in (3.18) w.r.t. time t and substituting back into the Bellman equation leads to an differential equation for π: First, we differentiate (3.18) w.r.t. time t and get

$$\frac{df}{dt} = f\,\frac{d\pi}{dt}\left\{\frac{\gamma}{\kappa}(1-\pi)^{-1} + \gamma\sigma^2(\eta-\gamma\sigma^2\pi)^{-1}\right\},$$

where we express the derivative in terms of f and the derivative of π. Substituting f and f_t into (3.17) and using the definition (2.3) yields the ODE (3.20) for π. □

Proof of Proposition 3.5. (i) In this case $\kappa_\alpha = \kappa$ and thus equation (3.22) becomes quadratic:

$$\pi(T)^2 + a\pi(T) + b = 0 \quad \text{with} \quad a \equiv -(m+1), \quad b \equiv m - \tfrac{1-\beta}{\beta}\beta^{\frac{1}{\kappa}}\tfrac{1}{\gamma\sigma^2}.$$

This equation has two real solutions, but one root is bigger than $0.5(m+1)$ and thus violates Proposition 3.2. Now, $C > 0$ and thus

$$\pi^*(T) = \frac{m+1}{2} - \sqrt{\frac{(m-1)^2}{4} + \frac{C}{\gamma\sigma^2}} < \frac{m+1}{2} - \frac{|m-1|}{2} = \min\{m, 1\}$$

Therefore, (3.24) is the correct solution and (i) follows.

(ii) follows since $\eta - \gamma\sigma^2\pi(T) = \gamma\sigma^2(1 - \pi(T))$. □

B Proof of Proposition 4.1

First, we need to establish some technical estimates that are summarized in the following lemma. To simplify notation, we set

$$\chi \equiv -\widetilde{K} = \frac{\kappa}{(\gamma-1)(1-\beta)} > 0.$$

Lemma B.1 (Solutions of Polynomial)**.** *We get the following representations:*

$$-0.5p = \frac{m + (1+m)\chi}{1+2\chi},$$

$$D = \frac{m^2}{1+2\chi} + \left(\frac{\chi(1-m)}{1+2\chi}\right)^2 + \frac{\frac{m}{\eta}(r - \frac{\delta}{1-\gamma})}{0.5(1+2\chi)}, \tag{B.44}$$

$$-0.5p - m = \frac{\chi(1-m)}{1+2\chi}. \tag{B.45}$$

Now assume that conditions (2.2)-(2.4) are satisfied. Then

$$-0.5p \in \left[\min\{m, \tfrac{1+m}{2}\}, \max\{m, \tfrac{1+m}{2}\}\right] \quad \textit{and} \quad D > 0.$$

Furthermore, if $m \le 1$ we get for the distance between $-0.5p$ and m

$$-0.5p - m \in [0, \sqrt{D}). \tag{B.46}$$

If $m > 1$ we obtain

$$-0.5p - \sqrt{D} < 1. \tag{B.47}$$

Proof of Lemma B.1. The representations follow from lengthy calculations. Furthermore, the function $\psi(\chi) = \frac{m+(1+m)\chi}{1+2\chi}$ is monotonic increasing for $m \leq 1$ (decreasing for $m > 1$), starting at m for $\chi = 0$, and having the limit $0.5(1+m)$ for $\chi \to \infty$. Besides, $D > 0$ since $\chi > 0$, and $r \geq -\delta/(\gamma-1)$. For $m \leq 1$ we get $-0.5p - m \geq 0$ since (B.45) and $\chi > 0$. Furthermore, $-0.5p-m < \sqrt{D}$ since the middle term in the representation of D is $(-0.5p-m)^2$ and the first term is strictly positive. To show (B.47) for $m > 1$, notice that

$$(-0.5p-1)^2 = \frac{(m-1)^2(1+\chi)^2}{(1+2\chi)^2}.$$

On the other hand,

$$D > \frac{m^2}{1+2\chi} + \left(\frac{\chi(1-m)}{1+2\chi}\right)^2 = \frac{m^2(1+2\chi)+\chi^2(1-m)^2}{(1+2\chi)^2}.$$

Now, the result follows since

$$(m-1)^2(1+\chi)^2 = (m-1)^2(1+2\chi) + \chi^2(m-1)^2 < m^2(1+2\chi) + \chi^2(1-m)^2.$$

□

Remark. Notice that $-0.5p - m < \sqrt{D}$ if $m \in (0,1]$, but $-0.5p - m = \sqrt{D}$ if $m = 0$. As already pointed out, the latter case is not interesting since then the agent puts all his wealth into the money market account in the standard portfolio problem anyway. Therefore, liquidity preferences are irrelevant for the portfolio decision. This trivial case is ruled out by assumption (2.3).

Now, we can provide the **proof of Proposition 4.1**. For the first root, we get $\pi_{01} = -0.5p + \sqrt{D} > \min\{m, \frac{1+m}{2}\}$, which violates the result of Proposition 3.2. Since (B.46) or (B.47), respectively, π_{02} is the only relevant solution to the first-order condition.

C Proofs of Section 5

Proof of Proposition 5.1. (i) follows from integrating the partial fraction expansion. (ii) follows from (4.28). (iii) and (iv) follow from the representation of $\mathcal{G}$ and (ii). Notice that for $m \in (0,1)$ the terms involving α_1 and α_3 are bounded, whereas for $m > 1$ the terms involving α_1 and α_4 are bounded. Finally, by construction, we have

$$\mathcal{G}'(\pi) = \frac{(1-\pi) + \frac{\gamma}{\kappa}(m-\pi)}{(\pi_{01}-\pi)(\pi^{\infty}-\pi)(1-\pi)(m-\pi)}.$$

The sign of $\mathcal{G}'$ is determined by the term $(\pi^{\infty}-\pi)$ since all other terms are positive and thus (v) follows. □

Proof of Proposition 5.2. Notice that

$$\frac{d\pi^*(t)}{dt} = \frac{d\mathcal{G}^{-1}}{dt}(\dots) \cdot (-\overline{K}).$$

Since $\overline{K} > 0$, the result follows by Proposition 5.1 (iv). Besides, π^* is well-defined because of Proposition 5.1 (v). □

Lemma C.1 (Inverse of $\mathcal{H}$)**.** *(i) If $\pi \in (-\infty, \pi^\infty)$, the inverse of $\mathcal{H}$ is given by*

$$\mathcal{H}^{-1}(y) = 1 - \sqrt{\frac{Dy}{y-1}}, \quad y > 1. \tag{C.48}$$

(ii) If $\pi \in (\pi^\infty, 1)$, the inverse of $\mathcal{H}$ is given by

$$\mathcal{H}^{-1}(y) = 1 - \sqrt{\frac{Dy}{y+1}}.$$

Proof of Lemma C.1. (i) In this case, we must invert the function

$$\mathcal{H}(\pi) = \frac{(1-\pi)^2}{(1-\sqrt{D}-\pi)(1+\sqrt{D}-\pi)}.$$

This yields two candidates

$$\mathcal{H}^{-1}(y) = 1 \pm \sqrt{\frac{Dy}{y-1}}, \quad y > 1.$$

We must have $\mathcal{H}^{-1}(y) \leq 1$, since $\pi^* \leq 1$. Therefore, (C.48) follows. Notice that $y > 1$ is not a restriction since the right-hand side of (5.35) is strictly greater than one. This is because $\mathcal{H}$ as a function of $\sqrt{D}$ is one for $\sqrt{D} = 0$, which is excluded by our assumptions, and strictly increasing in $\sqrt{D}$ for $\sqrt{D} > 0$. The case (ii) follows analogously to (i). □

Proof of Proposition 5.3. follows from (5.35) and Lemma C.1. □

D Proofs of Verification Results

Proof of Theorem 6.2. (i) follows from Proposition 5.2. (ii) follows from (3.18) and (3.19) since for our candidate π^* we have $\pi^* < \min\{m, 1\}$. In particular, there exist a constant $\bar{c}$ such that $c^* \leq \bar{c} X^{\pi^*, c^*}$. Since π^* is a smooth deterministic function on $[0, T]$, one can check that the pair (π^*, c^*) is admissible. □

Proof of Theorem 6.3. It is sufficient to show that for all admissible controls (π, c)

$$\int_0^T e^{-\delta s} \mathrm{E}[u(c_s, X_s^{\pi,c}(1-\pi_s))]\, ds + e^{-\delta T}\mathrm{E}[U(X_T^{\pi,c}, X_T^{\pi,c}(1-\pi_T))] \leq G(0, X_0)$$

and for our candidate (c^*, π^*)

$$\int_0^T e^{-\delta s} \mathrm{E}[u(c_s^*, X_s^*(1-\pi_s^*))]\, ds + e^{-\delta T}\mathrm{E}[U(X_T^*, X_T^*(1-\pi_T^*)] = G(0, X_0)$$

where $X^* \equiv X^{\pi^*,c^*}$.

First, notice that, by Proposition 3.1, our candidates π^* and c^* are the global maximizers of the Bellman equation (3.5) for the relevant region $(\pi, c) \in (-\infty, 1) \times (0, \infty)$. Therefore, our candidates satisfy the Bellman equation (without supremum) as equality, whereas all other admissible controls satisfy the equation as inequality.

Second, we like to stress that a function G satisfies the Bellman equation (3.5) if $V(t,x) \equiv G(t,x)e^{-\delta t}$ satisfies

$$0 = \sup_{\pi,c} \left\{ V_t + xrV_x + x\pi\eta V_x - cV_x + 0.5x^2\sigma^2\pi^2 V_{xx} + e^{-\delta t}u(c, x(1-\pi)) \right\} \tag{D.49}$$

with terminal condition $V(T,x) = e^{-\delta T}G(T,x)$. Notice that the maximizers of the Bellman equation are not affected by this transformation.

For an admissible control, we now apply Ito's formula to $V(t, X_t)$

$$\begin{aligned} dV &= [V_t + V_x X(r + \pi\eta - c/X) + 0.5V_{xx}X^2\sigma^2\pi^2]dt + V_x X\sigma\pi dW \\ &\overset{(D.49)}{=} -e^{-\delta t}u(c, X(1-\pi))dt + V_x X\sigma\pi dW \end{aligned}$$

where we omit all arguments. Evaluating at $t = T$ yields

$$V(T, X_T) + \int_0^T e^{-\delta t}u(c_t, X_t(1-\pi_t))dt \leq V(0, X_0) + \int_0^T V_x(t, X_t)X_t\sigma\pi_t dW_t, \tag{D.50}$$

where we have equality for (π^*, c^*). The integrand of the stochastic integral

$$V_x(t, X_t)X_t\sigma\pi_t = X_t^{1-\gamma}f^k(t)e^{-\delta t}\sigma\pi_t$$

is an $\mathcal{L}^2$-process since f is smooth and, by (iv) and (v) of an admissible strategy, π and the coefficients of X are bounded. Therefore, using the terminal condition and taking expectations yields

$$\mathrm{E}\Big[\int_0^T e^{-\delta t}u(c_t, X_t(1-\pi_t))dt + e^{-\delta T}U(X_T, X_T(1-\pi_T))\Big] \leq V(0, X_0)$$

with equality for (π^*, c^*), which gives the desired result. □

Proof of Theorem 6.4. We must show that

$$\int_0^\infty e^{-\delta t}\mathrm{E}[u(c_t, X_t^{\pi,c}(1-\pi_t))]\,dt \le G(X_0) \tag{D.51}$$

for all admissible strategies and

$$\int_0^\infty e^{-\delta t}\mathrm{E}[u(c^\infty, X_t^*(1-\pi^\infty))]\,dt = G(X_0) \tag{D.52}$$

for the optimal strategy where $X^* \equiv X^{\pi^\infty, c^\infty}$. As in the proof for the finite horizon, relation (D.50) still holds with $V(t,x) = G(x)e^{-\delta t}$. The integrand of the Ito integral is still an $\mathcal{L}^2$-process, i.e.

$$\mathrm{E}[V(T, X_T^{\pi,c})] + \mathrm{E}\Big[\int_0^T e^{-\delta t}u(c_t, X_t^{\pi,c}(1-\pi_t))dt\Big] \le V(0, X_0)$$

or in terms of G

$$e^{-\delta T}\mathrm{E}[G(X_T^{\pi,c})] + \mathrm{E}\Big[\int_0^T e^{-\delta t}u(c_t, X_t^{\pi,c}(1-\pi_t))dt\Big] \le G(X_0). \tag{D.53}$$

Since the previous inequality holds for all admissible strategies (π, c), we obtain

$$\sup_{\pi,c}\big(e^{-\delta T}\mathrm{E}[G(X_T^{\pi,c})]\big) + \mathrm{E}\Big[\int_0^T e^{-\delta t}u(c_t, X_t^{\pi,c}(1-\pi_t))dt\Big] \le G(X_0).$$

If we multiply the term involving the integral by -1, the term becomes positive and we can apply the monotonous convergence theorem. Therefore,

$$\lim_{T\to\infty}\mathrm{E}\Big[\int_0^T e^{-\delta t}u(c_t, X_t(1-\pi_t))dt\Big] = \mathrm{E}\Big[\int_0^\infty e^{-\delta t}u(c_t, X_t(1-\pi_t))dt\Big].$$

Hence, (D.51) follows if we can show that for every admissible strategy (π, c)

$$\lim_{T\to\infty}\left(e^{-\delta T}\sup_{\pi,c}\mathrm{E}[G(X_T)]\right) = 0$$

holds. Set $\theta \equiv \eta/\sigma$. Then

$$\begin{aligned} 0 \;&\ge\; \sup_{\pi,c}\big(e^{-\delta T}\mathrm{E}[G(X_T^{\pi,c})]\big) = \sup_{\pi,c}\left(\mathrm{E}\Big[\tfrac{1}{1-\gamma}(X_T^{\pi,c})^{1-\gamma}f^k\Big]\right)e^{-\delta T} \\ &\ge\; \underbrace{\sup_{\pi}\left(\mathrm{E}\Big[\tfrac{1}{1-\gamma}(X_T^{\pi,0})^{1-\gamma}\Big]\right)}_{(*)}e^{-\delta T}f^k = \underbrace{\tfrac{1}{1-\gamma}(X_0)^{1-\gamma}e^{(1-\gamma)(r+\frac{\theta^2}{2\gamma})T}}_{(**)}e^{-\delta T}f^k \longrightarrow 0 \text{ for } T\to\infty, \end{aligned}$$

since (2.4) holds and $\gamma > 1$. Notice that $(*)$ is the value function of an ordinary Merton problem with terminal wealth maximization only. One can check that it has the representation $(**)$.

Finally, we have to prove (D.52). Since (D.53) is satisfied as equality for the optimal strategy, it is sufficient to prove the transversality condition

$$\lim_{T\to\infty}\left(e^{-\delta T}\mathrm{E}[G(X_T^*)]\right)=0,$$

where

$$dX^* = X^*[(r+\eta\pi^\infty-\omega^\infty)dt+\sigma\pi^\infty dW]$$

and $\omega^\infty \equiv c^\infty/X^*$ is the consumption-wealth ratio which is given by (6.37). Therefore,

$$e^{-\delta T}\mathrm{E}[G(X_T^*)] = \mathrm{E}\Big[\tfrac{1}{1-\gamma}(X_T^*)^{1-\gamma}f^k\Big]e^{-\delta T} = \tfrac{1}{1-\gamma}X_0^{1-\gamma}e^{\varphi T}f^k$$

where $\varphi \equiv (1-\gamma)(r+\eta\pi^\infty-\omega^\infty-0.5\gamma\sigma^2(\pi^\infty)^2)-\delta$ is a constant. It remains to show that

$$\varphi < 0. \tag{D.54}$$

By (3.21), the optimal stock demand π^∞ satisfies the equation

$$\frac{P_2(\pi)}{\widetilde{K}} = -\frac{1-\beta}{\frac{1}{\gamma-1}+\beta}\Big(r-\frac{\delta}{1-\gamma}+(\eta-0.5\gamma\sigma^2\pi)\pi\Big)+(\eta-\gamma\sigma^2\pi)(1-\pi)=0$$

Therefore,

$$\begin{aligned}
\varphi &= (1-\gamma)\left(r-\frac{\delta}{1-\gamma}+\eta\pi^\infty-0.5\gamma\sigma^2(\pi^\infty)^2-\frac{\beta}{1-\beta}(\eta-\gamma\sigma^2\pi^\infty)(1-\pi^\infty)\right)\\
&= \frac{(\gamma-1)\beta}{1-\beta}\Bigg(-\frac{1-\beta}{\beta}\underbrace{\left[r-\frac{\delta}{1-\gamma}+(\eta-0.5\gamma\sigma^2\pi^\infty)\pi^\infty\right]}_{(\times)}+(\eta-\gamma\sigma^2\pi^\infty)(1-\pi^\infty)\Bigg)\\
&< \frac{(\gamma-1)\beta}{1-\beta}\frac{P_2(\pi^\infty)}{\widetilde{K}}=0,
\end{aligned}$$

which shows (D.54). Notice that $(\times)$ is strictly positive since we assume (2.4) and $\pi^\infty > 0$. □

E Recursive Utility

Using (7.40) we again eliminate consumption from the Bellman equation (7.39):

$$0=\sup_{\pi}\left\{G_t+xrG_x+x\pi\eta G_x+0.5x^2\sigma^2\pi^2G_{xx}-\theta\delta G+K_\phi(x(1-\pi))^{\frac{(\beta-1)(\phi-1)}{\kappa_\phi}}G_x^{\frac{\beta(\phi-1)}{\kappa_\phi}}((1-\gamma)G)^{\frac{\theta-1}{\theta\kappa_\phi}}\right\}$$

with

$$K_\phi \equiv \frac{1}{1-\phi}\beta^{\frac{\beta(1-\phi)}{\kappa_\phi}}-\beta^{\frac{1}{\kappa_\phi}}.$$

Substituting the guess (3.10) into the Bellman equation yields

$$0 = \tfrac{k}{1-\gamma} f_t + \Big(r - \tfrac{\delta}{1-\phi} + \pi\eta - 0.5\gamma\sigma^2\pi^2 \Big) f + K_\phi (1-\pi)^{\frac{(\beta-1)(\phi-1)}{\kappa_\phi}} f^{1-\frac{k}{\theta\kappa_\phi}},$$

where we assume that the Bellman equation (E.55) is evaluated at the optimal stock demand $\pi = \pi^*$. Using (7.42), we obtain

$$0 = \tfrac{k}{1-\gamma} f_t + \Big(r - \tfrac{\delta}{1-\phi} + \pi\eta - 0.5\gamma\sigma^2\pi^2 + \widetilde{K}_\phi (\eta - \gamma\sigma^2\pi)(1-\pi) \Big) f \tag{E.55}$$

with

$$\widetilde{K}_\phi \equiv K_\phi \frac{\beta}{1-\beta} \beta^{-\frac{1}{\kappa_\phi}} = \frac{\kappa_\phi}{(1-\phi)(1-\beta)}.$$

By choosing

$$k = \theta\kappa_\phi = \theta(1 + \beta(\phi - 1)),$$

one can isolate f in (7.42):

$$f = \frac{\frac{1-\beta}{\beta} \beta^{\frac{1}{\kappa_\phi}}}{(\eta - \gamma\sigma^2\pi)(1-\pi)^{\frac{\phi}{\kappa_\phi}}} \tag{E.56}$$

As before the optimal stock demand determines the value function. Furthermore, by substituting (E.56) into (7.41), the consumption-wealth ratio is again given by (3.19). Taking derivatives in (E.56) w.r.t. time t yields

$$\frac{df}{dt} = f \, \frac{d\pi}{dt} \left\{ \frac{\phi}{\kappa_\phi} (1-\pi)^{-1} + \gamma\sigma^2 (\eta - \gamma\sigma^2\pi)^{-1} \right\},$$

where we express the derivative in terms of f and the derivative of π. Substituting back into the Bellman equation leads to the differential equation (7.43) for π. Since f must satisfy the terminal condition (3.11), the terminal condition for π can be calculated from (7.42) by solving for $\pi(T)$ in the following equation

$$\eta - \gamma\sigma^2\pi(T) = C_\phi \Big(1 - \pi(T) \Big)^{-\frac{\kappa_{\phi,\alpha}}{\kappa_\phi}} \tag{E.57}$$

with $\kappa_{\phi,\alpha} \equiv 1 + \alpha(\phi - 1)$ and

$$C_\phi \equiv \frac{1-\beta}{\beta} \beta^{\frac{1}{\kappa_\phi}}.$$

As before there this condition can be solved explicitly for $\alpha = \beta$ and $m = 1$.

Table 1: **Constants.** This table summarizes relevant constants that are defined in the paper.

Name		Definition	On Page
m	$\equiv$	$\frac{\eta}{\gamma\sigma^2}$	3
κ	$\equiv$	$1+\beta(\gamma-1)$	4
K	$\equiv$	$\beta^{\frac{1}{\kappa}-1}\frac{\kappa}{1-\gamma}$	5
$\widetilde{K}$	$\equiv$	$K\frac{\beta}{1-\beta}\beta^{-\frac{1}{\kappa}} = \frac{\kappa}{(1-\gamma)(1-\beta)}$	6
κ_α	$\equiv$	$1+\alpha(\gamma-1)$	8
C	$\equiv$	$\frac{1-\beta}{\beta}\beta^{\frac{1}{\kappa}}$	8
a	$\equiv$	$-(m+1)$	23
b	$\equiv$	$m-\frac{1-\beta}{\beta}\beta^{\frac{1}{\kappa}}\frac{1}{\gamma\sigma^2}$	23
$\widehat{K}$	$\equiv$	$\gamma\sigma^2(\widetilde{K}-0.5)$	9
p	$\equiv$	$\frac{\eta-\widetilde{K}(\eta+\gamma\sigma^2)}{\widehat{K}}$	9
q	$\equiv$	$\frac{r-\frac{\delta}{1-\gamma}+\widetilde{K}\eta}{\widehat{K}}$	9
D	$\equiv$	$p^2/4-q$	9
χ	$\equiv$	$-\widetilde{K} = \frac{\kappa}{(\gamma-1)(1-\beta)}$	23
π^∞	$\equiv$	$\pi_{02} = -0.5p-\sqrt{D}$	9
$\overline{K}$	$\equiv$	$-\widehat{K}\frac{\gamma-1}{k}$	11
α_1	$\equiv$	$\frac{1+\frac{\gamma}{k}m-\pi_{01}(1+\frac{\gamma}{k})}{(\pi_{01}-\pi^\infty)(1-\pi_{01})(\pi_{01}-m)}$	11
α_2	$\equiv$	$\frac{1+\frac{\gamma}{k}m-\pi^\infty(1+\frac{\gamma}{k})}{(\pi_{01}-\pi^\infty)(1-\pi^\infty)(m-\pi^\infty)}$	11
α_3	$\equiv$	$\frac{\frac{\gamma}{k}}{(1-\pi_{01})(1-\pi^\infty)}$	11
α_4	$\equiv$	$\frac{1}{(\pi_{01}-m)(m-\pi^\infty)}$	11

Table 2: **Baseline parameter values.** This table reports the baseline parameters.

Symbol	Meaning	Value
r	Interest rate	0.01
η	Equity premium	0.05
σ	Stock volatility	0.17
γ	Risk aversion	3
δ	Time preference rate	0.03
β	Consumption weight	0.99
α	Consumption weight at death	1.00

Figure 1: **The function $\mathcal{G}(\pi)$ in the baseline case.** The figure depicts the function $\mathcal{G}(\pi)$. Baseline parameter values are used yielding a Merton result of $m = 0.5767$. Besides, the terminal condition for the finite-horizon solution is $\pi^*(T) = 0.3867$. The solution to the infinite-horizon problem $\pi^\infty = 0.5696$ is the location of the pole of $\mathcal{G}$.

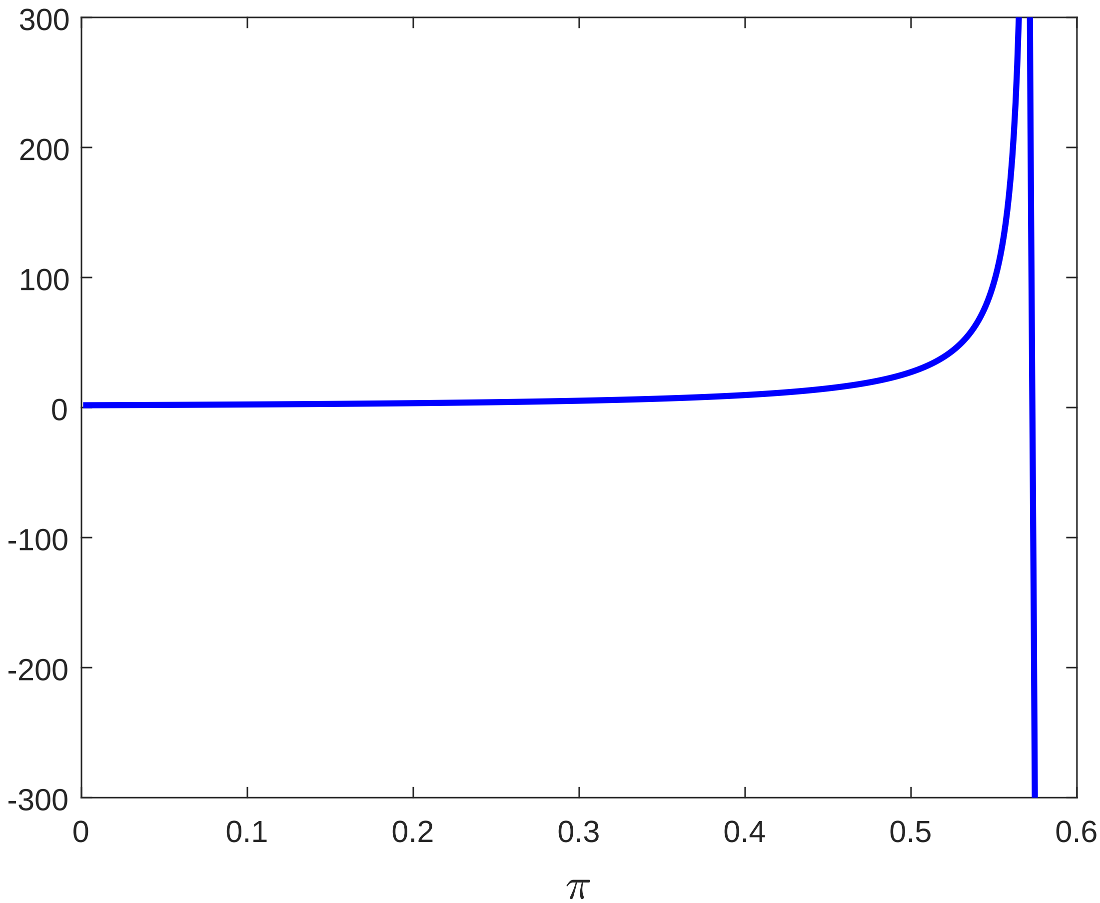

Figure 2: **Optimal investments over the life cycle in the baseline case.** The figure depicts the optimal proportion invested in stocks for the benchmark calibration reported in Table 2. It also shows the solution $m = 0.5767$ to an ordinary Merton problem and the solution $\pi^{\infty} = 0.5696$ to the problem with infinite horizon. The terminal condition for the finite-horizon solution is $\pi^*(T) = 0.3867$.

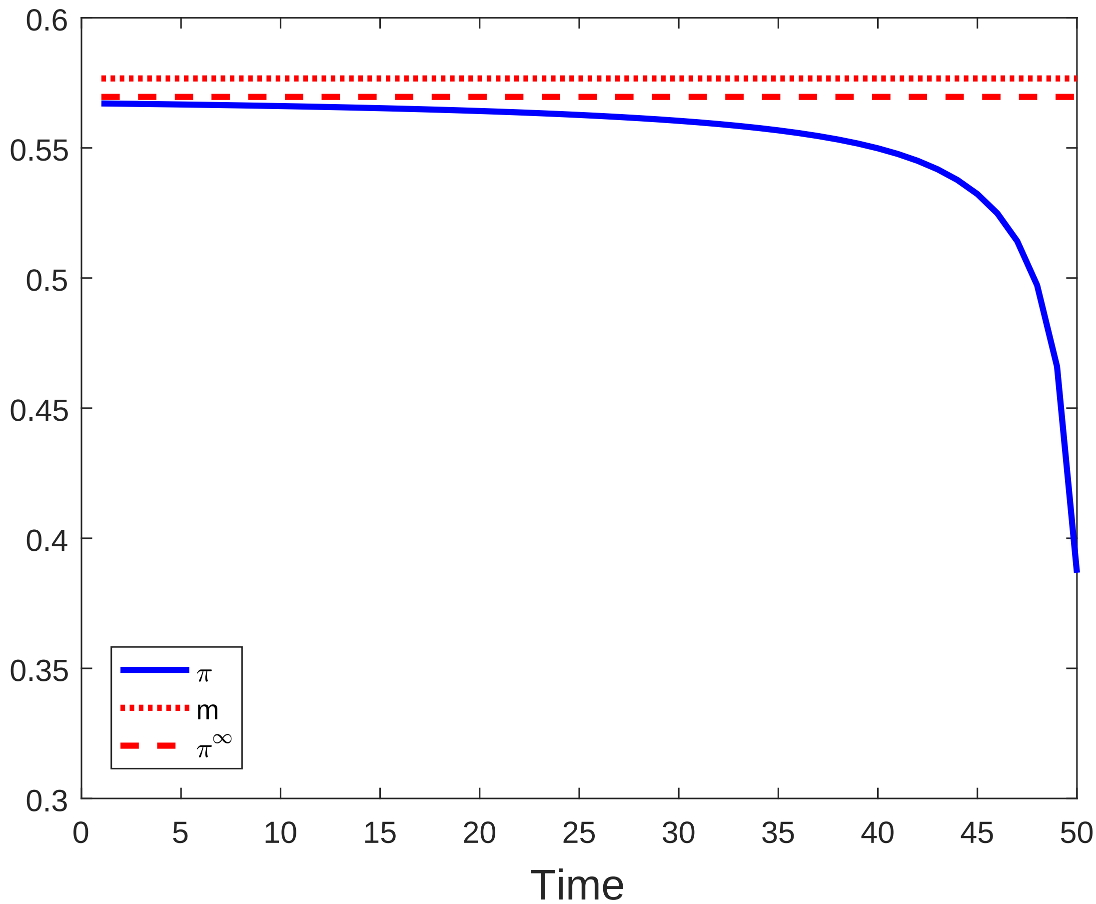

Figure 3: **Optimal investments over the life cycle for $\boldsymbol{m = 1}$.** The figure depicts the optimal proportion invested in stocks for $m = 1$. In this case, we increase the equity risk premium from $\eta = 0.05$ in the benchmark case to $\eta = 0.0867$. For the remaining parameters the baseline values are used. The solution to the infinite-horizon problem is $\pi^{\infty} = 0.9274$. The terminal condition for the finite-horizon solution is $\pi^{*}(T) = 0.6580$.

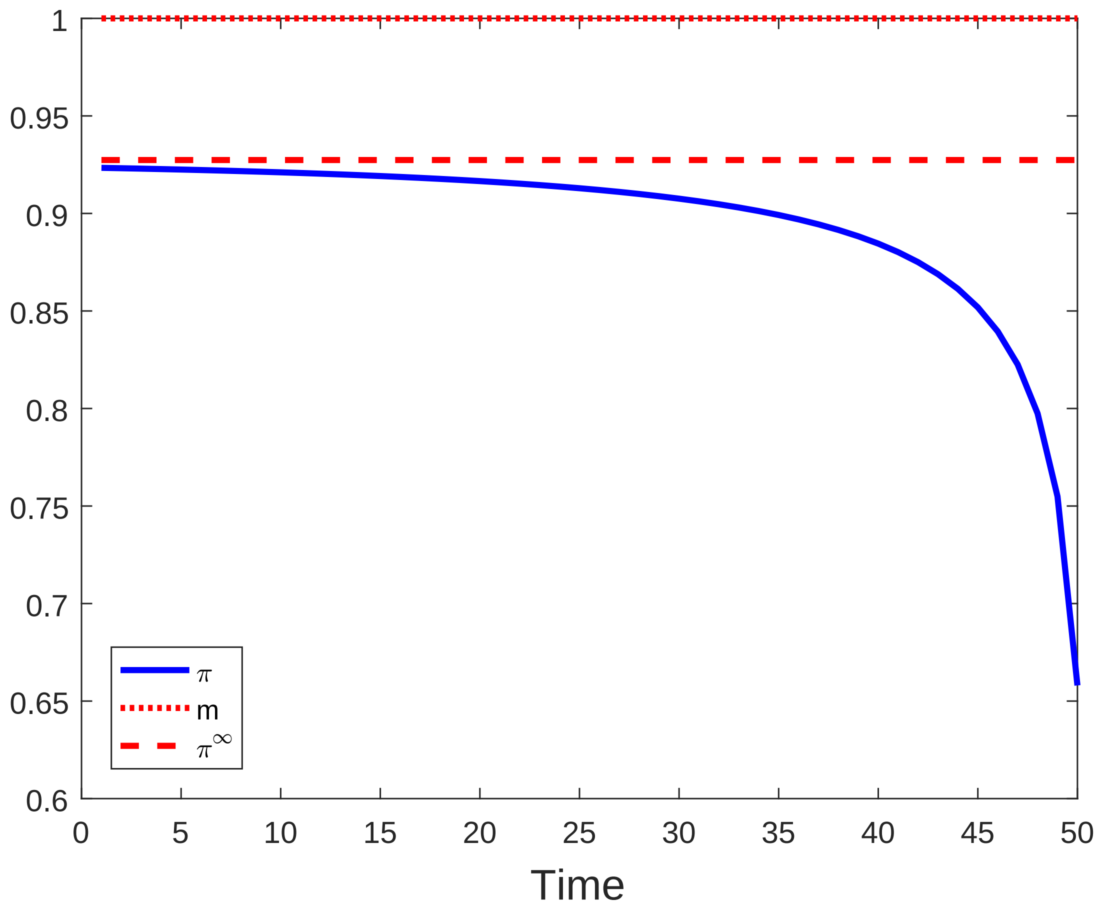

Figure 4: **Optimal investments over the life cycle for $m > 1$.** The figure depicts the optimal proportion invested in stocks for $m > 1$. In this case, we increase the equity risk premium from $\eta = 0.05$ in the benchmark case to $\eta = 0.09537$ so that $m = 1.1$. For the remaining parameters the baseline values are used. The solution to the infinite-horizon problem is $\pi^{\infty} = 0.9582$. The terminal condition for the finite-horizon solution is $\pi^{*}(T) = 0.7042$.

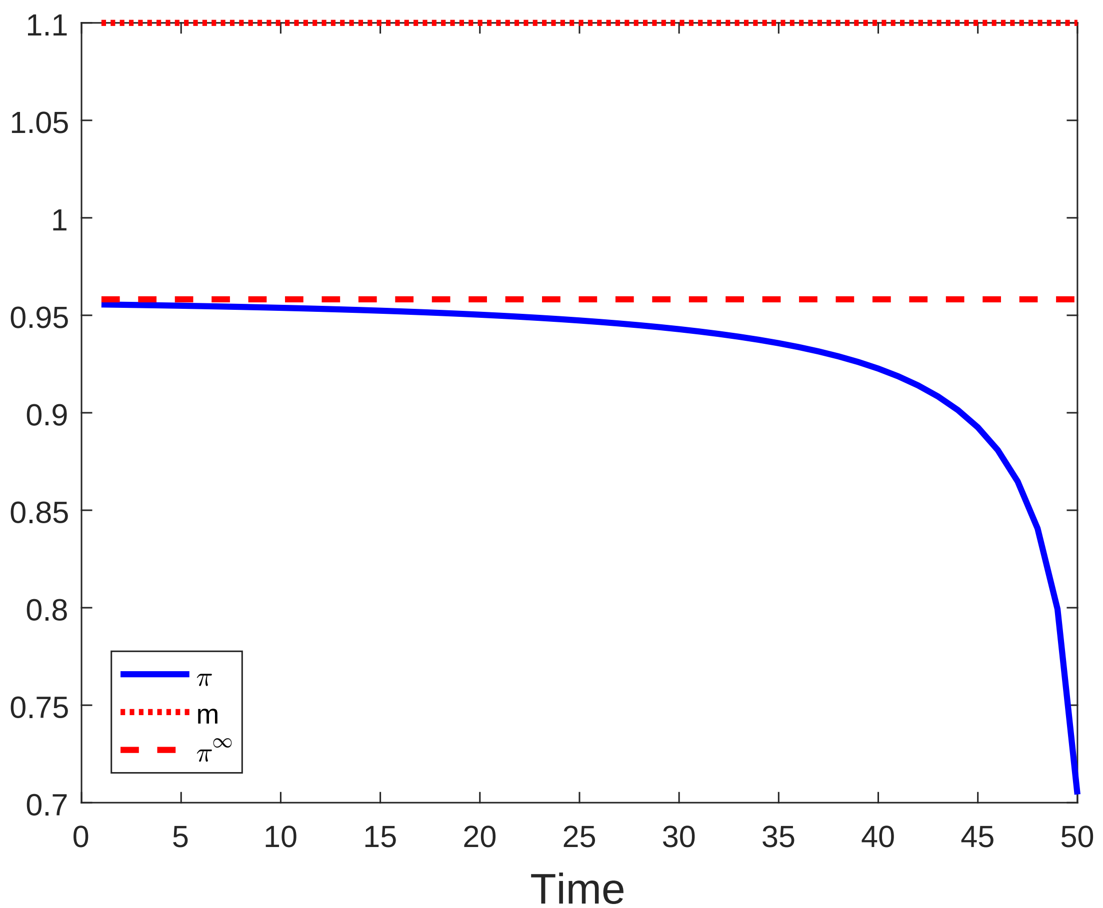

Figure 5: **Optimal investments over the life cycle for different $\boldsymbol{\beta}$.** The figure depicts the optimal proportion invested in stocks for different values of β. For the remaining parameters the baseline values are used. The terminal conditions for the finite-horizon solutions are $\pi^*(T) = 0.3867$ for $\beta = 0.99$, $\pi^*(T) = 0.2596$ for $\beta = 0.98$, $\pi^*(T) = 0.1566$ for $\beta = 0.97$, and $\pi^*(T) = 0.0675$ for $\beta = 0.96$.

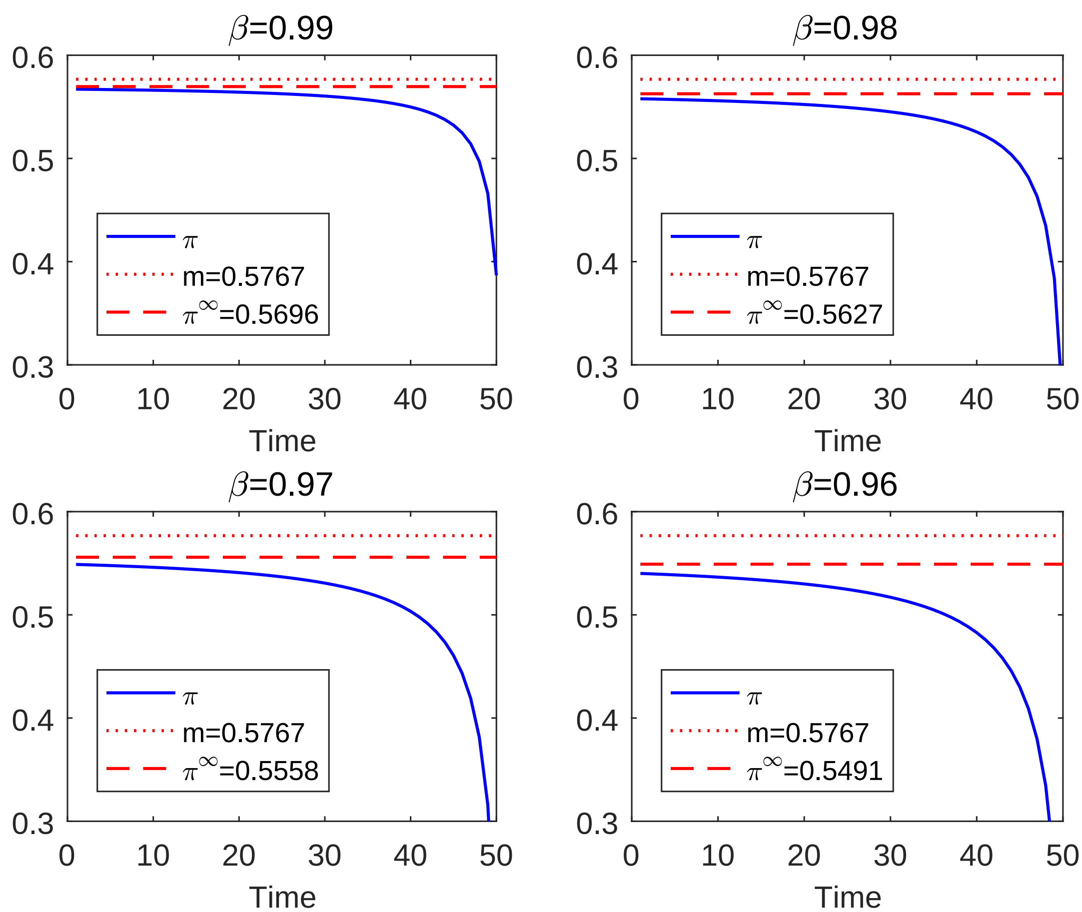

Figure 6: **Optimal investments over the life cycle for different γ.** The figure depicts the optimal proportion invested in stocks for different values of γ. For the remaining parameters the baseline values are used. The terminal conditions for the finite-horizon solutions are $\pi^*(T) = 0.5888$ for $\gamma = 1.5$, $\pi^*(T) = 0.5094$ for $\gamma = 2$, $\pi^*(T) = 0.3867$ for $\gamma = 3$, and $\pi^*(T) = 0.3065$ for $\gamma = 4$.

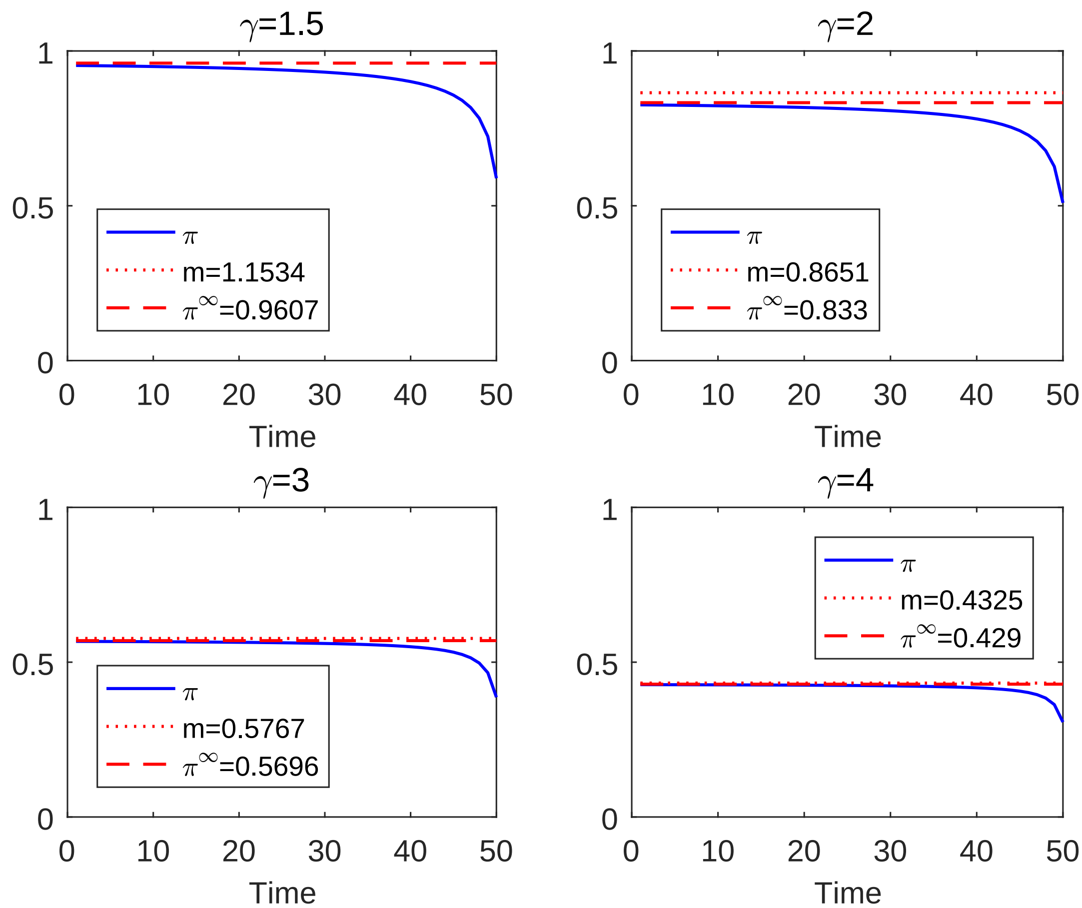

Figure 7: **Optimal investments over the life cycle with recursive utility and different ϕ.** This figure assumes that the agent has recursive utility as introduced in Section 7. Therefore, we can vary the elasticity of intertemporal substitution (EIS) independently of the risk aversion. The figure depicts the optimal proportion invested in stocks for different values of $\phi = 1/\text{EIS}$. For the remaining parameters the baseline values are used. The terminal conditions for the finite-horizon solutions are $\pi^*(T) = 0.3903$ for $\phi = 0.5$, $\pi^*(T) = 0.3871$ for $\phi = 2$, $\pi^*(T) = 0.3867$ for $\phi = 3$, and $\pi^*(T) = 0.3866$ for $\phi = 4$.

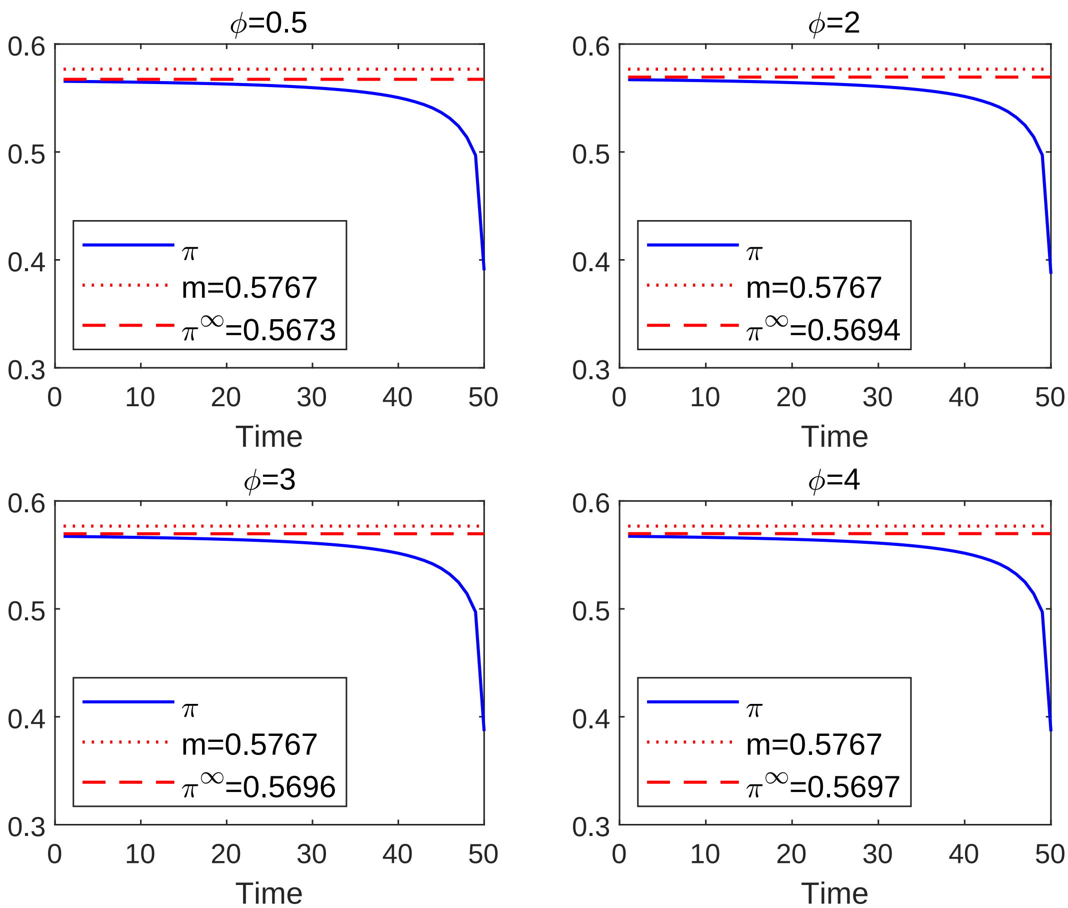

Figure 8: **Optimal investments over the life cycle with recursive utility and different γ.** This figure assumes that the agent has recursive utility as introduced in Section 7. Since we use recursive utility, we can now vary the risk aversion without changing the elasticity of intertemporal substitution (EIS). The figure depicts the optimal proportion invested in stocks for different values of γ and fixed $\phi = 3$, which is the benchmark value in the time-additive case because there we have $\gamma = \phi$. For the remaining parameters the baseline values are used. The terminal conditions for the finite-horizon solutions are $\pi^*(T) = 0.5873$ for $\gamma = 1.5$, $\pi^*(T) = 0.5088$ for $\gamma = 2$, $\pi^*(T) = 0.3867$ for $\gamma = 3$, and $\pi^*(T) = 0.3066$ for $\gamma = 4$.