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Optimal Consumption and Investment with Epstein-Zin Recursive Utility

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Non-Technical Summary

Decision making of agents is described by utility functionals. For instance, in the representative agent model, that has dominated macroeconomics for the last three decades, there is one individual, the representative agent, whose preferences have to be described. In the classical version, the agent is assumed to have a time-separable von Neumann-Morgenstern utility function and to have access to a financial market that is complete. Both these specifications are potential sources of why the classical framework is not able to explain several empirical facts about asset prices. Economists have responded to these challenges by assuming that agents have more general preferences (e.g. recursive preferences) and/or by postulating more involved asset or endowment processes (e.g. jumps, disasters, unspanned diffusion) that lead to incomplete financial markets.

To calculate the values of cash flows in such a representative agent economy, the stochastic discount factor is the key ingredient. A stochastic discount factor induces a pricing rule that determines all asset prices in an economy. It is thus of crucial importance that the agent's utility can be described in a tractable way. In every continuous-time model, this utility satisfies a certain partial differential equation that can be derived by applying a stochastic representation theorem for expectations. For recursive preferences, this can be reduced to an equation belonging to a particular class of semi-linear partial differential equations. Such equations are inherently difficult to solve and, in general, it is not even clear whether they admit (unique smooth) solutions. So far, researchers have usually resorted to approximation techniques of unclear precision (e.g. Campbell-Shiller approximation for non-unit elasticity of intertemporal substitution) and considered affine frameworks.

In this context, our paper makes a significant contribution: For possibly non-affine models, we prove the existence of a solution and develop a fast and accurate numerical method to compute this solution. Our scheme solves the nonlinear partial differential equation by iteratively solving certain linear partial differential equations. We also derive worst-case bounds for the accuracy of our methodology. Therefore, our results provide a solid basis for future research in asset pricing with recursive preferences.

Furthermore, we also contribute to the extensive literature on dynamic incomplete-market portfolio theory. This area is concerned with an agent's consumption-portfolio choice problem where returns are not necessarily independent and identically distributed. We study an incomplete-market consumption-portfolio problem that nests several classical frameworks. In contrast to the existing literature, we do not restrict our analysis to affine models but allow for recursive preferences. We reduce the Bellman equation to a partial differential equation that belongs to the same class as the above-mentioned equation in asset pricing. Researchers have so far relied on approximative methods in this context as well, although in general not even the issue of existence has been resolved. Therefore, as an additional contribution of this article, we provide a verification theorem demonstrating that a suitable smooth solution of the reduced Bellman equation is also the solution to the consumption-portfolio problem. Following the

same agenda as outlined above, we then establish existence of a solution and construct this solution by fixed point arguments. Again, our numerical method provides a fast and accurate way of calculating the investor's indirect utility and optimal strategies. Our results thus also establish a tractable approach to incomplete-market consumption-portfolio choice problems with recursive preferences.

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Optimal Consumption and Investment with Epstein-Zin Recursive Utility

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Abstract We study continuous-time optimal consumption and investment with Epstein-Zin recursive preferences in incomplete markets. We develop a novel approach that rigorously constructs the solution of the associated Hamilton-Jacobi-Bellman equation by a fixed point argument and makes it possible to compute both indirect utility and, more importantly, optimal strategies. Based on these results, we also establish a fast and accurate method for numerical computations. Our setting is not restricted to affine asset price dynamics; we only require boundedness of the underlying model coefficients.

Keywords consumption-portfolio choice · asset pricing · stochastic differential utility · incomplete markets · fixed point approach · FBSDE

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1 Introduction

This article contributes to the extensive literature on dynamic incomplete-market portfolio theory, a research area that according to Cochrane [14] is at the same time "important" and "hard". We study a class of incomplete-market consumption-portfolio problem that nests (suitably truncated[1] versions of) classical frameworks such as Kim and Omberg [24], Campbell and Viceira [8], Barberis [2], Wachter [41], Chacko and Viceira [12] and Liu [30], among others. With the exception of [12], these authors assume time-additive CRRA preferences. We consider Epstein-Zin recursive utility, which contains time-additive CRRA utility as a special case. The Hamilton-Jacobi-Bellman (short: Bellman or HJB) equation of the associated consumption-portfolio problem can be transformed into a semilinear partial differential equation. Nonlinear equations of this class are challenging and, in general, it is not even clear whether they admit unique smooth solutions. Moreover, in the absence of affine dynamics, closed-form solutions of the HJB equation are not available. Thus researchers so far have been forced to resort to linearization techniques of unclear precision such as, e.g., the Campbell-Shiller approximation in models with affine dynamics. In an extension of the framework of [26], this paper provides – for possibly non-affine models and assuming only suitable boundedness conditions – existence and uniqueness of solutions to such equations. Moreover, we develop a fast and accurate numerical method for the computation of both indirect utility and, even more importantly, optimal strategies. Our results thus establish a tractable and constructive approach to incomplete-market consumption-portfolio problems with Epstein-Zin recursive utility in continuous time.

From a mathematical perspective, our contributions can be summarized as follows: First, we establish a verification theorem which demonstrates that a suitable $C^{1,2}$ solution of the reduced HJB equation is the value function of the consumption-portfolio problem. The proof is based on a combination of dynamic programming arguments and utility gradient inequalities for recursive utility. Second, we provide an explicit construction of such a $C^{1,2}$ solution based on fixed point arguments for the associated system of forward-backward stochastic differential equations. We study the Feynman-Kac representation mapping Φ that is associated with a power transform of the HJB equation and obtain a fixed point in the space of continuous functions as a limit of iterations of Φ. Using the probabilistic representation of this solution we are able to improve this to convergence in $C^{0,1}$. This not only yields a theoretical convergence result, but also leads directly to a numerical method with superexponential speed of convergence that allows us to determine optimal strategies efficiently via iteratively solving linear partial differential equations.

[1] Our analysis imposes no structural conditions on the underlying model coefficients, but requires them to be bounded; see (A1) and (A2) in Section 4 and (A1') in Section 7.

The remainder of the paper is structured as follows: Section 2 discusses the related literature. Section 3 introduces the agent's utility functional for continuous-time recursive preferences. In Section 4 we formulate the consumption-portfolio problem, establish existence of a solution of the associated HJB equation, and derive candidate optimal strategies. Section 5 provides a verification result which demonstrates that the candidate solutions are indeed optimal. Section 6 provides the proof of our main existence and uniqueness result (Theorem 4.6). Building on this, Section 7 sets the basis for our numerical method, which is illustrated in Section 8. Section 8 also offers an informal user's guide on how to apply our theoretical results to concrete problems. The Appendix contains proofs and auxiliary results.

2 Links to the Literature

This article is related to several strands of research. First, we contribute to the literature on dynamic incomplete-market portfolio theory. Liu [30] considers portfolio problems with unspanned risk and time-additive utility. The framework of Liu [30] already nests a number of popular models, including those of Kim and Omberg [24], Campbell and Viceira [8], Barberis [2], and Wachter [41], as special cases. Given appropriate boundedness conditions on the underlying coefficients, our approach can be used to generalize several of his solutions to settings where asset price dynamics are non-affine or non-quadratic and where the agent has recursive utility. Recursive utility has been developed by Kreps and Porteus [27,28], Epstein and Zin [22] and Duffie and Epstein [17]. Chacko and Viceira [12] study a consumption-portfolio problem with affine stochastic volatility and recursive preferences. They find an explicit solution for unit elasticity of intertemporal substitution (EIS) and approximate the solution for non-unit EIS using the Campbell-Shiller technique. Our approach makes it possible to extend their analysis to problems with non-affine specifications of stochastic volatility, without having to rely on a priori approximations. Finally, our results are also related to Schroder and Skiadas [36], who focus on complete markets, and to Schroder and Skiadas [37], who provide necessary and sufficient optimality conditions in a general homothetic setting by duality methods and obtain explicit solutions for unit EIS.

Second, the mathematical analysis of this article has ties to a number of articles. The verification argument used to solve the consumption-portfolio problem builds on the so-called utility gradient approach that has been developed in a series of papers by Duffie, Schroder, and Skiadas including [19], [36], [37], [38] and [39]; see also Seiferling and Seifried [40]. We generalize the verification results in Duffie and Epstein [17], who derive a verification result for aggregators satisfying a Lipschitz condition, and of [26], who consider Epstein-Zin preferences under parameter restrictions. Our results are also related to the findings of Duffie and Lions [18], who study the existence of stochastic differential utility using PDE methods, and to Marinacci and Montrucchio [34], who establish existence and uniqueness of recursive utility in discrete time. The analysis of Berdjane and Pergamenshchikov [3] is based on a fixed point argument related to the one we use in this paper, but is focused on the special case where the agent has time-additive utility with risk aversion below unity and where the state process has constant volatility. In a recent paper that appeared after this article had been finished, Xing [42] addresses a closely related class of portfolio optimization problems using BSDE techniques. [42] complements the analysis of this paper: It requires weaker boundedness (rsp., integrability) conditions,[2] but does not provide information on how to determine optimal strategies. In addition, the analysis of [42] is restricted to the case when both relative risk aversion and EIS are greater than one. This article provides a constructive, fast and accurate numerical methodology (substantiated by a rigorous convergence analysis) to find both indirect utility and, more importantly, optimal strategies.

Finally, our analysis of existence and uniqueness also contributes to the literature on semilinear partial differential equations (PDEs) and backward and forward-backward stochastic differential equations (BSDEs and FBSDEs, respectively). We demonstrate that the FBSDE associated with the semilinear PDE that is relevant for our applications in consumption-portfolio choice admits a unique bounded solution. Importantly, the driver of this FBSDE is not Lipschitz, so standard results do not apply. We thus contribute to the growing literature on non-Lipschitz BSDEs and FBSDEs, including, among others, Kobylanski [25], Briand and Carmona [5], Briand and Hu [6], and Delbaen, Hu and Richou [16]. In addition, by deriving an associated Feynman-Kac representation, this paper adds to the literature that connects FBSDEs to semilinear Cauchy problems; see, e.g., Pardoux and Peng [35], Delarue [15] and Ma, Yin and Zhang [33] and the references therein.

3 Consumption Plans and Epstein-Zin Preferences

We fix a probability space $(\Omega, \mathcal{F}, P)$ with a complete right-continuous filtration $(\mathcal{F}_t)_{t\in[0,T]}$ that is generated by a Wiener process $(W, \bar{W})$. We denote the *consumption space* by $\mathfrak{C} \triangleq (0,\infty)$. In the following, we are interested in an agent's preferences on the space of dynamic consumption plans.

Definition 3.1 (Consumption Plans) A progressively measurable process c with values in $\mathfrak{C}$ is a *consumption plan* if

$$c \in \mathcal{C} \triangleq \left\{ c \in \mathcal{D}^+ \,:\, \mathrm{E}\left[\int_0^T c_t^p \,\mathrm{d}\,t + c_T^p\right] < \infty \text{ for all } p \in \mathbb{R} \right\}.$$

Here we denote the set of square-integrable progressive processes by

$$\mathcal{D} \triangleq \left\{ X = (X_t)_{t\in[0,T]} \text{ progressively measurable} \,:\, \mathrm{E}\left[\int_0^T X_t^2 \,\mathrm{d}\,t + X_T^2\right] < \infty \right\}$$

and write $\mathcal{D}^+ \triangleq \{X \in \mathcal{D} \,:\, X_t > 0 \text{ for } t \in [0,T]\}$ for its strictly positive cone.

[2] In particular, [42] covers specifications with (untruncated) affine dynamics as in Kim and Omberg [24] and Heston [23].

The agent's preferences on $\mathcal{C}$ are described by a utility index $\nu : \mathcal{C} \to \mathbb{R}$. Thus

$$c \in \mathcal{C} \text{ is weakly preferred to } \bar{c} \in \mathcal{C} \quad \text{if and only if} \quad \nu(c) \geq \nu(\bar{c}),$$

see [17], [22] and [40]. To construct the Epstein-Zin utility index, let

$$\delta > 0, \quad \gamma > 0, \quad \psi > 0 \quad \text{with } \gamma, \psi \neq 1.$$

be given and put $\phi \triangleq \frac{1}{\psi}$. If $\gamma < 1$ set $\mathfrak{U} \triangleq (0, \infty)$ and for $\gamma > 1$ set $\mathfrak{U} \triangleq (-\infty, 0)$. Then the continuous-time Epstein-Zin aggregator is given by $f : \mathfrak{C} \times \mathfrak{U} \to \mathbb{R}$,

$$f(c, v) \triangleq \delta\theta v \left[\left(\frac{c}{((1-\gamma)v)^{\frac{1}{1-\gamma}}} \right)^{1-\frac{1}{\psi}} - 1 \right] \quad \text{where } \theta \triangleq \frac{1-\gamma}{1-\phi} \in \mathbb{R}, \ \theta \neq 0.$$

Here γ represents the agent's relative risk aversion, ψ is his elasticity of intertemporal substitution (EIS) and δ is his rate of time preference. Moreover we define $U : \mathfrak{C} \to \mathbb{R}$, $U(x) \triangleq \varepsilon^{1-\gamma} \frac{1}{1-\gamma} x^{1-\gamma}$ as a CRRA utility function for bequest where $\varepsilon \in (0, \infty)$ is a weight factor. For Epstein-Zin utility to be well-defined, throughout this article we impose the following assumptions:

(E1) For every $c \in \mathcal{C}$ there exists a unique semimartingale V^c such that $\mathrm{E}[\sup_{t \in [0,T]} |V_t^c|^p] < \infty$ for all $p \in \mathbb{R}$ and

$$V_t^c = \mathrm{E}_t \left[\int_t^T f(c_s, V_s^c) \, \mathrm{d}\, s + U(c_T) \right] \quad \text{for all } t \in [0, T].$$

(E2) For every $\bar{c} \in \mathcal{C}$ we have the *utility gradient inequality*

$$V_0^c \leq V_0^{\bar{c}} + \langle m(\bar{c}), c - \bar{c} \rangle \quad \text{for all } c \in \mathcal{C}$$

where the inner product on $\mathcal{D}$ is given by $\langle X, Y \rangle = \mathrm{E}[\int_0^T X_t Y_t \, \mathrm{d}\, t + X_T Y_T]$ and the utility gradient at $\bar{c}$ is defined via

$$m_t(\bar{c}) \triangleq \exp \left(\int_0^t f_v(\bar{c}_s, V_s^{\bar{c}}) \, \mathrm{d}\, s \right) \nabla_t(\bar{c})$$

with $\nabla_t(\bar{c}) \triangleq f_c(\bar{c}_t, V_t^{\bar{c}})$ for $t \in [0, T)$ and $\nabla_T(\bar{c}) \triangleq U'(\bar{c}_T)$.

It is shown in [40] that (E1) and (E2) are both satisfied if either

$$\gamma\psi, \psi \geq 1 \quad \text{or alternatively} \quad \gamma\psi, \psi \leq 1,^3 \tag{3.1}$$

but we wish to stress that the analysis of this article applies to all parametrizations of Epstein-Zin utility for which (E1) and (E2) can be verified. This leads to the following definition:

Definition 3.2 (Utility Index) The Epstein-Zin utility index $\nu : \mathcal{C} \to \mathfrak{U}$ is given by $\nu(c) \triangleq V_0^c$ where V^c is the unique process given by (E1).

[3] Condition (3.1) holds if and only if one of conditions (a), (b), (c) and (d) in Proposition 3.2 of [26] is satisfied; see also (2) in [40]. We are not aware of rigorous results that ensure (E1) and (E2) for parametrizations not subsumed by (3.1).

The classical time-additive utility specification

$$\nu(c) = \mathrm{E}\left[\int_0^T e^{-\delta s} u(c_s)\,\mathrm{d}\,s + e^{-\delta T} U(c_T)\right]$$

where $u:\ \mathfrak{C} \to \mathbb{R}$, $u(x) \triangleq \frac{1}{1-\gamma} x^{1-\gamma}$, is subsumed as the special case of the Epstein-Zin parametrization where $\gamma = \phi$. Note that (3.1) covers all additive utility specifications. Hence the analysis of this article applies in particular to consumption-portfolio optimization with additive CRRA preferences and arbitrary risk aversion parameter $\gamma \neq 1$. In addition, (3.1) subsumes all parametrizations with $\gamma, \psi \geq 1$.

Remark. The specifications $\gamma = 1$ or $\psi = 1$ correspond to unit relative risk aversion or unit EIS, respectively; $\gamma = \psi = 1$ represents time-additive logarithmic utility. The case of unit EIS, $\psi = 1$, is well-understood and has been studied extensively in the literature; see, e.g., Schroder and Skiadas [37] and Chacko and Viceira [12]. The analysis of this article applies *mutatis mutandis* to these special cases. ◇

4 Consumption-Portfolio Selection with Epstein-Zin Preferences

4.1 Financial Market Model

Two securities are traded. The first is a locally risk-free asset (e.g., a money market account) M with dynamics

$$\mathrm{d}\,M_t = r(Y_t) M_t\,\mathrm{d}\,t,$$

while the second asset (e.g., a stock or stock index) S is risky and satisfies

$$\mathrm{d}\,S_t = S_t\left[(r + \lambda(Y_t))\,\mathrm{d}\,t + \sigma(Y_t)\,\mathrm{d}\,W_t\right].$$

The interest rate $r:\ \mathbb{R} \to \mathbb{R}$ and the stock's excess return and volatility $\lambda, \sigma : \mathbb{R} \to \mathbb{R}$ are assumed to be measurable functions of a state process Y with dynamics

$$\mathrm{d}\,Y_t = \alpha(Y_t)\,\mathrm{d}\,t + \beta(Y_t)\left(\rho\,\mathrm{d}\,W_t + \sqrt{1-\rho^2}\,\mathrm{d}\,\bar{W}_t\right), \quad Y_0 = y.$$

Here $\alpha, \beta : \mathbb{R} \to \mathbb{R}$ are measurable functions and $\rho \in [-1, 1]$ denotes the correlation between stock returns and the state process. Throughout this article, we assume:

(A1) The coefficients $r, \lambda, \sigma, \alpha$ are bounded and Lipschitz continuous; the coefficient β is bounded and has a bounded Lipschitz continuous derivative.

(A2) Ellipticity condition: $\inf_{y\in\mathbb{R}} \sigma(y) > 0$ and $\inf_{y\in\mathbb{R}} \beta(y) > 0$.

The investor's wealth dynamics are given by

$$\mathrm{d}\, X_t^{\pi,c} = X_t^{\pi,c}\left[(r(Y_t) + \pi_t\lambda(Y_t))\,\mathrm{d}\, t + \pi_t\sigma(Y_t)\,\mathrm{d}\, W_t\right] - c_t\,\mathrm{d}\, t, \quad X_0 = x \tag{4.1}$$

where π_t denotes the fraction of wealth invested in the risky asset at time t, the constant $x > 0$ is the investor's initial wealth and c his consumption plan.

Definition 4.1 (Admissible Strategies) The pair of strategies (π, c) is *admissible* for initial wealth $x > 0$ if it belongs to the set

$$\mathcal{A}(x) \triangleq \{(\pi, c) \in \mathcal{D} \times \mathcal{C} \,:\, X_t^{\pi,c} > 0 \text{ for all } t \in [0, T] \text{ and } c_T = X_T^{\pi,c}\}\,.$$

Since the investor's preferences are described by a recursive utility functional of Epstein-Zin type, an admissible pair $(\pi, c) \in \mathcal{A}(x)$ yields utility

$$\nu(c) \triangleq V_0^c, \quad \text{where } V_t^c \triangleq \mathrm{E}_t\left[\int_t^T f(c_s, V_s^c)\,\mathrm{d}\, s + U(X_T^{\pi,c})\right] \text{ for } t \in [0, T].$$

Definition 4.2 (Consumption-Portfolio Problem) Given initial wealth $x > 0$, the investor's *consumption-portfolio problem* is to maximize utility over the class of admissible strategies $\mathcal{A}(x)$,

$$\text{find} \quad (\pi^\star, c^\star) \in \mathcal{A}(x) \quad \text{such that} \quad \nu(c^\star) = \sup_{(\pi,c)\in\mathcal{A}(x)} \nu(c). \tag{P}$$

Remark. Problem (P) has been widely studied in the literature: Schroder and Skiadas [36] investigate the case of complete markets. Schroder and Skiadas [37–39] provide necessary and sufficient optimality conditions for general homothetic and translation-invariant preferences. Moreover, [37] solve the consumption-portfolio problem for an investor with unit EIS in closed form. Chacko and Viceira [12] obtain closed-form solutions for an investor with unit EIS in an inverse Heston stochastic volatility model, and Kraft, Seifried and Steffensen [26] derive explicit solutions for a non-unit EIS investor whose preference parameters satisfy the condition

$$\psi = 2 - \gamma + \tfrac{(1-\gamma)^2}{\gamma}\rho^2. \tag{H}$$

Berdjane and Pergamenshchikov [3] study the above-described consumption-portfolio problem in the special case where the investor has additive preferences with relative risk aversion $\gamma \in (0, 1)$. Figure 4.1 depicts the parametrizations for which solutions are known in the literature. Xing [42] focuses on the quadrant northeast of $\gamma = 1$, $\psi = 1$. ◇

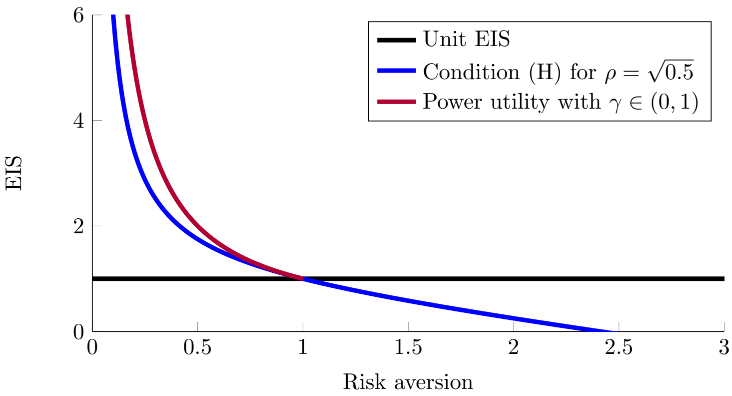

Fig. 4.1 Known Solutions. This figure depicts combinations of risk aversion γ and elasticity of intertemporal substitution ψ for which solutions to consumption-portfolio problems with unspanned risk are known.

4.2 The HJB Equation

We consider the dynamic programming equation associated with problem (P),

$$0 = \sup_{\pi\in\mathbb{R},\, c\in(0,\infty)} \Big\{ w_t + x(r+\pi\lambda)w_x - cw_x + \tfrac{1}{2}x^2\pi^2\sigma^2 w_{xx} + \alpha w_y + \tfrac{1}{2}\beta^2 w_{yy} + x\pi\sigma\beta\rho w_{xy} + f(c,w)\Big\} \tag{4.2}$$

subject to the boundary condition $w(T,x,y) = \varepsilon^{1-\gamma}\frac{1}{1-\gamma}x^{1-\gamma}$. Following Zariphopoulou [43] we conjecture a solution of the form

$$w(t,x,y) = \tfrac{1}{1-\gamma}x^{1-\gamma}h(t,y)^k, \quad (t,x,y) \in [0,T]\times(0,\infty)\times\mathbb{R} \tag{4.3}$$

where k is a constant and $h \in C^{1,2}([0,T]\times\mathbb{R})$ is strictly positive with $h(T,\cdot\,) = \hat{\varepsilon} \triangleq \varepsilon^{\frac{1-\gamma}{k}}$. Choosing $k \triangleq \frac{\gamma}{\gamma+(1-\gamma)\rho^2}$ and solving the first-order conditions leads to the following definition:

Definition 4.3 The *candidate optimal strategies* are given by

$$\hat{\pi} \triangleq \frac{\lambda}{\gamma\sigma^2} + \frac{k}{\gamma}\frac{\beta\rho}{\sigma}\frac{h_y}{h} \quad \text{and} \quad \hat{c} \triangleq \delta^\psi h^{q-1}x \tag{4.4}$$

where $q \in \mathbb{R}$, $q \neq 1$ is given by

$$q \triangleq 1 - \frac{\psi k}{\theta}$$

and where h is a strictly positive solution of the semilinear partial differential equation (PDE)

$$0 = h_t - \tilde{r}h + \tilde{\alpha}h_y + \tfrac{1}{2}\beta^2 h_{yy} + \tfrac{\delta^\psi}{1-q}h^q, \quad h(T,\cdot\,) = \hat{\varepsilon} \tag{4.5}$$

with $\tilde{r} \triangleq -\frac{1}{k}\left[r(1-\gamma)+\frac{1}{2}\frac{1-\gamma}{\gamma}\frac{\lambda^2}{\sigma^2}-\delta\theta\right]$ and $\tilde{\alpha} \triangleq \alpha+\frac{1-\gamma}{\gamma}\frac{\lambda\beta\rho}{\sigma}$. In the following we refer to (4.5) as the *reduced HJB equation.*

Remark. The function h in (4.3) is closely related to the candidate for the agent's optimal consumption-wealth ratio as used in, e.g., [7], [9], and [12]. More precisely, by (4.4) we have $\frac{\hat{c}}{x}=\delta^{\psi}h^{-\frac{\psi k}{\theta}}$ so we can represent the candidate for the value function equivalently as $w(t,x,y)=\frac{1}{1-\gamma}x^{1-\gamma}\delta^{\theta}\left(\frac{\hat{c}}{x}\right)^{-\frac{\theta}{\psi}}$. ◇

Lemma 4.4 *If $h \in C^{1,2}([0,T]\times\mathbb{R})$ is a strictly positive solution of* (4.5)*, then the function given by $w(t,x,y)=\frac{1}{1-\gamma}x^{1-\gamma}h(t,y)^k$ solves the HJB equation* (4.2)*.*

Lemma 4.5 *The functions $\tilde{r}$ and $\tilde{\alpha}$ are bounded and Lipschitz continuous.*

Remark. Note that for all $\rho \in [-1,1]$ we have

$$q = 1-\frac{1-\phi}{1-\gamma}C \quad \text{where} \quad C \triangleq \frac{\psi\gamma}{\gamma(1-\rho^2)+\rho^2} > 0.$$

Thus $q<1$ if and only if $\frac{1-\phi}{1-\gamma}>0$ and $q>1$ if and only if $\frac{1-\phi}{1-\gamma}<0$; see Table 4.1 and Figure 4.2. ◇

$q<1$	$q=1$	$q>1$
$\frac{1-\phi}{1-\gamma}>0$	$\phi=1$	$\frac{1-\phi}{1-\gamma}<0$

Table 4.1 Ranges of q. This table reports the range of the exponent q in (4.5) depending on the risk aversion γ and the reciprocal of the elasticity of intertemporal substitution ϕ.

We now state a general existence result for the semilinear PDE (4.5):

Theorem 4.6 *For all $\gamma,\psi,\delta>0$ with $\gamma,\psi\neq 1$ there exist a solution $h \in C^{1,2}([0,T]\times\mathbb{R})$ to the reduced HJB equation* (4.5) *and positive constants $0<\underline{h}<\overline{h}$ such that*

$$\underline{h} \le h \le \overline{h} \quad \text{and} \quad \|h_y\|_\infty < \infty. \tag{4.6}$$

Moreover, h is the unique solution of (4.5) *that is bounded above and away from* 0*.*

Theorem 4.6 is one of the main results of this article. A large part of Section 6 below is dedicated to its proof. Before that, we demonstrate in Section 5 how Theorem 4.6 is fundamental for the solutions of consumption-portfolio choice problems.

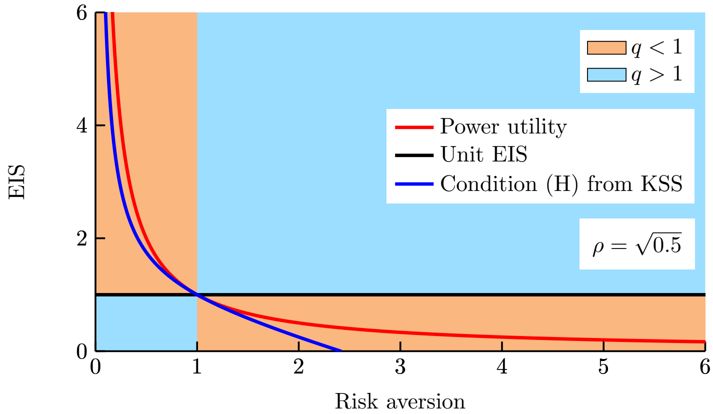

Fig. 4.2 Range of *q*. This figure depicts the range of the exponent q in (4.5) depending on the risk aversion γ and the elasticity of intertemporal substitution ψ. Condition (H) is calculated for $\rho = \sqrt{0.5}$.

5 Verification

In this section we establish the following verification result:

Theorem 5.1 (Solution of the Consumption-Portfolio Problem (P)**)** *Let h be a solution to the reduced HJB equation* (4.5) *as in Theorem 4.6. Then the corresponding candidate strategies* $(\hat{\pi}, \hat{c})$*,*

$$\hat{\pi}_t = \frac{\lambda(Y_t)}{\gamma\sigma(Y_t)^2} + \frac{k}{\gamma}\frac{\beta(Y_t)\rho}{\sigma(Y_t)}\frac{h_y(t,Y_t)}{h(t,Y_t)}\,, \quad \hat{c}_t = \delta^{\psi} h(t,Y_t)^{q-1} X_t^{\hat{\pi},\hat{c}} \quad \textit{for } t \in [0,T) \tag{5.1}$$

and $\hat{c}_T \triangleq X_T^{\hat{\pi},\hat{c}}$ *are optimal for the consumption-portfolio problem* (P)*.*

Here we slightly abuse notation by setting $\hat{\pi}_t = \hat{\pi}(t, Y_t)$ and $\hat{c}_t \triangleq \hat{c}(t, X_t^{\hat{\pi},\hat{c}}, Y_t)$ for $t \in [0,T)$. This will not give rise to confusion in the following.

5.1 Abstract Utility Gradient Approach

Let $(\bar{\pi}, \bar{c}) \in \mathcal{A}(x)$ be a given fixed consumption-portfolio strategy (below we take the candidate solution in (5.1), but the general argument here does not rely on that specific choice), and let $\bar{m} \triangleq m(\bar{c})$ denote the utility gradient at $\bar{c}$, see condition (E2). For every strategy $(\pi, c) \in \mathcal{A}(x)$ define the *deflated wealth process* $Z^{\pi,c}$ via

$$Z_t^{\pi,c} \triangleq \bar{m}_t X_t^{\pi,c} + \int_0^t \bar{m}_s c_s \,\mathrm{d}\, s \quad \text{for } t \in [0,T].$$

Then we have the following general verification theorem:

Theorem 5.2 (Abstract Verification) *Suppose that for every admissible strategy* $(\pi, c) \in \mathcal{A}(x)$ *the deflated wealth process* $Z^{\pi,c}$ *is a local martingale, and that* $Z^{\bar{\pi},\bar{c}}$ *is a true martingale. Then* $(\bar{\pi}, \bar{c})$ *is optimal for the consumption-portfolio problem* (P)*.*

Proof The utility gradient inequality (E2) evaluated at $\bar{c}$ implies

$$V_0^c \le V_0^{\bar{c}} + \langle \bar{m}, c - \bar{c} \rangle = V_0^{\bar{c}} + \mathrm{E}\left[\int_0^T \bar{m}_s (c_s - \bar{c}_s)\,\mathrm{d}\,s + \bar{m}_T (X_T^{\pi,c} - X_T^{\bar{\pi},\bar{c}})\right]$$

where

$$\int_0^T \bar{m}_s (c_s - \bar{c}_s)\,\mathrm{d}\,s + \bar{m}_T (X_T^{\pi,c} - X_T^{\bar{\pi},\bar{c}}) = Z_T^{\pi,c} - Z_T^{\bar{\pi},\bar{c}}.$$

Here the process $Z^{\pi,c}$ is a positive local martingale, hence a supermartingale, while $Z^{\bar{\pi},\bar{c}}$ is a martingale by assumption. Since $X_0^{\pi,c} = X_0^{\bar{\pi},\bar{c}} = x$, we obtain

$$\mathrm{E}[Z_T^{\pi,c} - Z_T^{\bar{\pi},\bar{c}}] \le \mathrm{E}[Z_0^{\pi,c} - Z_0^{\bar{\pi},\bar{c}}] = f_c(\bar{c}_0, V_0^{\bar{c}})(X_0^{\pi,c} - X_0^{\bar{\pi},\bar{c}}) = 0. \qquad \square$$

5.2 Admissibility of the Candidate Solution $(\hat{\pi}, \hat{c})$

In the proof of Theorem 5.1 below, we apply the abstract verification result in Theorem 5.2 to the candidate $(\hat{\pi}, \hat{c})$ in (5.1). Therefore in the following we verify that the conditions of Theorem 5.2 are satisfied for that strategy.

We first establish admissibility of $(\hat{\pi}, \hat{c})$. Thus suppose that h is the solution of the reduced HJB equation (4.5) provided by Theorem 4.6, and let $(\hat{\pi}, \hat{c})$ be given by (5.1). For simplicity of notation we write

$$\hat{V} \triangleq V^{\hat{c}}, \qquad \hat{X} \triangleq X^{\hat{\pi},\hat{c}}, \qquad \hat{m} \triangleq m(\hat{c})$$

for the utility process, the wealth process and the utility gradient associated with $(\hat{\pi}, \hat{c})$. The proofs of the following results are deferred to Appendix A.

Lemma 5.3 *The candidate optimal wealth process has all moments, i.e.*

$$\mathrm{E}\big[\sup_{t\in[0,T]} \hat{X}_t^p\big] < \infty \quad \textit{for all } p \in \mathbb{R}.$$

In particular $\hat{X}_t > 0$ *for all* $t \in [0, T]$ *a.s.*

As a consequence we can show that $\hat{c} \in \mathcal{C}$ and $\hat{V}_t = w(t, \hat{X}_t, Y_t)$, where by Lemma 4.4 the function $w(t, x, y) \triangleq \frac{1}{1-\gamma} x^{1-\gamma} h(t, y)^k$ solves the HJB equation (4.2):

Lemma 5.4 (Admissibility of $\hat{c}$) *Let* $V_t \triangleq w(t, \hat{X}_t, Y_t)$, $t \in [0, T]$*. Then* $V = \hat{V}$ *and* $w_x(t, \hat{X}_t, Y_t) = f_c(\hat{c}_t, \hat{V}_t)$*. Moreover we have*

$$\mathrm{E}\big[\sup_{t\in[0,T]} |\hat{c}_t|^p\big] < \infty \quad \textit{for all } p \in \mathbb{R}$$

and in particular $\hat{c} \in \mathcal{C}$*.*

Combining Lemmas 5.3 and 5.4 it follows in particular that $(\hat{\pi}, \hat{c}) \in \mathcal{A}(x)$.

5.3 Optimality of the Candidate Solution

Next we show that the deflated wealth process $Z^{\pi,c}$ is a local martingale for every admissible consumption-portfolio strategy $(\pi, c) \in \mathcal{A}(x)$. The proofs can again be found in Appendix A.

Lemma 5.5 (Dynamics of $Z^{\pi,c}$) *For all $(\pi, c) \in \mathcal{A}(x)$ the deflated wealth process $Z^{\pi,c}$ is a local martingale with dynamics*

$$\mathrm{d}\, Z_t^{\pi,c} = \hat{m}_t X_t^{\pi,c} \left[\left(\pi_t \sigma(Y_t) - \frac{\lambda(Y_t)}{\sigma(Y_t)} \right) \mathrm{d}\, W_t + k\sqrt{1-\rho^2}\beta(Y_t) \frac{h_y(t, Y_t)}{h(t, Y_t)} \,\mathrm{d}\, \bar{W}_t \right].$$

For the candidate optimal process $(\hat{\pi}, \hat{c})$ this implies

$$\begin{aligned} \mathrm{d}\, Z_t^{\hat{\pi},\hat{c}} = \hat{m}_t \hat{X}_t \Big[& \left(\tfrac{1-\gamma}{\gamma} \tfrac{\lambda(Y_t)}{\sigma(Y_t)} + \tfrac{k}{\gamma} \beta(Y_t) \rho \tfrac{h_y(t,Y_t)}{h(t,Y_t)} \right) \mathrm{d}\, W_t \\ & + k\sqrt{1-\rho^2}\beta(Y_t) \tfrac{h_y(t,Y_t)}{h(t,Y_t)} \,\mathrm{d}\, \bar{W}_t \Big]. \end{aligned}$$

Lemma 5.6 *The process $Z^{\hat{\pi},\hat{c}}$ is a martingale.*

Combining the preceding results, we can complete the

Proof (of Theorem 5.1) By Lemmas 5.5 and 5.6 the conditions of Theorem 5.2 are fulfilled. Thus Theorem 5.2 implies that $(\hat{\pi}, \hat{c})$ is optimal for the consumption-portfolio problem (P). □

6 Feynman-Kac Fixed Point Approach to the HJB Equation

The goal of this section is to prove Theorem 4.6. We present a constructive method to obtain a classical solution of the reduced HJB equation (4.5),

$$0 = h_t - \tilde{r}h + \tilde{\alpha}h_y + \tfrac{1}{2}\beta^2 h_{yy} + \tfrac{\delta^\psi}{1-q} h^q, \quad h(T, \cdot) = \varepsilon^{\frac{1-\gamma}{k}} = \hat{\varepsilon}.$$

To this end, we study the following forward-backward stochastic differential equation (FBSDE) that is associated with the reduced HJB equation:

$$\mathrm{d}\, \eta_t^{t_0,y_0} = \tilde{\alpha}(\eta_t^{t_0,y_0}) \,\mathrm{d}\, t + \beta(\eta_t^{t_0,y_0}) \,\mathrm{d}\, W_t, \quad \eta_{t_0}^{t_0,y_0} = y_0 \tag{6.1}$$

$$\mathrm{d}\, X_t^{t_0,y_0} = -\left[\tfrac{\delta^\psi}{1-q} (X_t^{t_0,y_0})^q - \tilde{r}(\eta_t^{t_0,y_0}) X_t^{t_0,y_0} \right] \mathrm{d}\, t + Z_t^{t_0,y_0} \,\mathrm{d}\, W_t, \quad X_T^{t_0,y_0} = \hat{\varepsilon} \tag{6.2}$$

where $t_0 \in [0, T]$ and $y_0 \in \mathbb{R}$. We will demonstrate that there exists a unique family $(X^{t,y})_{t\in[0,T]}^{y\in\mathbb{R}}$ of bounded positive solutions to this FBSDE system, and that this yields a solution to the reduced HJB equation via the generalized Feynman-Kac formula

$$h(t,y) = X_t^{t,y} = \mathrm{E}_t\left[\int_t^T e^{-\int_t^s \tilde r(\eta_\tau^{t,y})\,\mathrm{d}\,\tau}\tfrac{\delta^\psi}{1-q}(X_s^{t,y})^q\,\mathrm{d}\,s + \hat\varepsilon e^{-\int_t^T \tilde r(\eta_\tau^{t,y})\,\mathrm{d}\,\tau}\right].$$

Remark. In this context, a natural way to think of the function h is as the fixed point of the *Feynman-Kac operator* $\Phi:\ C_b([0,T]\times\mathbb{R})\to C_b([0,T]\times\mathbb{R})$,

$$(\Phi h)(t,y) \triangleq \mathrm{E}_t\left[\int_t^T e^{-\int_t^s \tilde r(\eta_\tau^{t,y})\,\mathrm{d}\,\tau}\tfrac{\delta^\psi}{1-q}h(s,\eta_s^{t,y})^q\,\mathrm{d}\,s + \hat\varepsilon e^{-\int_t^T \tilde r(\eta_\tau^{t,y})\,\mathrm{d}\,\tau}\right].$$

In Section 7 we elaborate this perspective in detail. ◇

The connection between semilinear PDEs and (F)BSDEs is well-established in the mathematical literature. While classical results, including Pardoux and Peng [35], Ma, Protter and Young [32] and Ma, Yin and Zhang [33], impose a Lipschitz condition on the generator, recent research has focused on relaxing that assumption. Starting from Kobylanski [25], existence and uniqueness results for BSDEs with quadratic and convex drivers have been obtained. Thus Briand and Carmona [5], Delarue [15], Briand and Hu [6] and Delbaen, Hu and Richou [16] replace the Lipschitz assumption by a so-called monotonicity condition, while retaining a polynomial growth condition. In general, however, the driver in the FBSDE system (6.1), (6.2) is neither Lipschitz, nor does it satisfy monotonicity or polynomial growth conditions. Hence, results from this literature cannot be applied to that equation. By establishing suitable a priori estimates for (6.1), (6.2) and (4.5), we prove the relevant existence, uniqueness and representation results in the following.

6.1 Solving the FBSDE System: A Fixed Point Approach

Until further notice, we fix $t_0\in[0,T]$ and $y_0\in\mathbb{R}$ and let $\eta\triangleq\eta^{t_0,y_0}$ be given by (6.1). For a progressively measurable process $(X)_{t\in[t_0,T]}$ we write

$$\|X\|_\infty = \operatorname{ess\,sup}_{\mathrm{d}\,t\otimes P}|X_t|$$

and denote by $\mathcal{D}_\infty$ the collection of all progressively measurable processes $(X_t)_{t\in[t_0,T]}$ with $\|X\|_\infty<\infty$. Clearly $(\mathcal{D}_\infty,\|\cdot\|_\infty)$ forms a Banach space. In the following we construct a fixed point of the operator $\Psi:\mathcal{D}_b\to\mathcal{D}_\infty$, $X\mapsto\Psi X$ defined via

$$(\Psi X)_t \triangleq \mathrm{E}_t\left[\int_t^T e^{-\int_t^s \tilde r(\eta_\tau)\,\mathrm{d}\,\tau}\tfrac{\delta^\psi}{1-q}(0\vee X_s)^q\,\mathrm{d}\,s + \hat\varepsilon e^{-\int_t^T \tilde r(\eta_\tau)\,\mathrm{d}\,\tau}\right] \tag{6.3}$$

where $\mathcal{D}_b\triangleq\{X\in\mathcal{D}_\infty : (0\vee X)^q \text{ is well-defined and bounded}\}$.

Remark. For every $X\in\mathcal{D}_b$ the process ΨX is continuous and thus has a progressive modification. Indeed, setting

$$M_t \triangleq \mathrm{E}_t\left[\int_0^T e^{-\int_0^s \tilde r(\eta_\tau)\,\mathrm{d}\,\tau}\tfrac{\delta^\psi}{1-q}(0\vee X_s)^q\,\mathrm{d}\,s + \hat\varepsilon e^{-\int_0^T \tilde r(\eta_\tau)\,\mathrm{d}\,\tau}\right]$$

we have that M is a bounded continuous martingale and

$$(\Psi X)_t = e^{\int_0^t \tilde{r}(\eta_\tau)\,\mathrm{d}\,\tau} M_t - \int_0^t e^{-\int_t^s \tilde{r}(\eta_\tau)\,\mathrm{d}\,\tau} \tfrac{\delta^\psi}{1-q}(0 \vee X_s)^q \,\mathrm{d}\, s.$$

In the following we always work with that version of ΨX. ◇

Lemma 6.1 *Let $X \in \mathcal{D}_b$ with $\Psi X = X$. Then X solves the BSDE*[4]

$$\mathrm{d}\, X_t = -\left[\tfrac{\delta^\psi}{1-q}(0 \vee X_t)^q - \tilde{r}(\eta_t)X_t\right]\mathrm{d}\, t + Z_t \,\mathrm{d}\, W_t, \quad X_T = \hat{\varepsilon}. \tag{6.4}$$

In particular, if X is positive then it is a solution of (6.2).

Proof Let $X \in \mathcal{D}_\infty$ with $\Psi X = X$ and set

$$Y_t \triangleq e^{-\int_{t_0}^t \tilde{r}(\eta_\tau)\,\mathrm{d}\,\tau} X_t = \mathrm{E}_t\left[\int_t^T e^{-\int_{t_0}^s \tilde{r}(\eta_\tau)\,\mathrm{d}\,\tau} \tfrac{\delta^\psi}{1-q}(0 \vee X_s)^q \,\mathrm{d}\, s + \hat{\varepsilon} e^{-\int_{t_0}^T \tilde{r}(\eta_\tau)\,\mathrm{d}\,\tau}\right]$$

and

$$M_t \triangleq \mathrm{E}_t\left[\int_{t_0}^T e^{-\int_{t_0}^s \tilde{r}(\eta_\tau)\,\mathrm{d}\,\tau} \tfrac{\delta^\psi}{1-q}(0 \vee X_s)^q \,\mathrm{d}\, s + \hat{\varepsilon} e^{-\int_{t_0}^T \tilde{r}(\eta_\tau)\,\mathrm{d}\,\tau}\right].$$

Then M is a bounded martingale and we have

$$Y_t = M_t - \int_{t_0}^t e^{-\int_{t_0}^s \tilde{r}(\eta_\tau)\,\mathrm{d}\,\tau} \tfrac{\delta^\psi}{1-q}(0 \vee X_s)^q \,\mathrm{d}\, s.$$

With integration by parts it follows that X solves (6.4). If X is positive then $X = 0 \vee X$ and thus X also solves (6.2). □

Our construction of a fixed point of Ψ is based on the following ramification of the classical Banach fixed point argument for the space $\mathcal{D}_\infty$:

Proposition 6.2 (Fixed Point Iteration in $\mathcal{D}_\infty$) *Let $S : A \to A$ be an operator on a closed, non-empty subset A of $\mathcal{D}_\infty$ and assume that there are constants $c > 0$, $\varrho \geq 0$ such that for all $X, Y \in A$ we have a Lipschitz condition of the form*

$$|(SX)_t - (SY)_t| \leq c\int_t^T \mathrm{E}_t\left[e^{(s-t)\varrho}|X_s - Y_s|\right]\mathrm{d}\, s \quad \textit{a.s. for all } t \in [t_0, T].$$

Then S has a unique fixed point. Moreover, the iterative sequence $X_{(n)} \triangleq SX_{(n-1)}$ $(n = 1, 2, \dots)$ with an arbitrarily chosen $X_{(0)} \in A$ satisfies

$$\|X_{(n)} - X\|_\infty \leq e^{T\varrho}(\|X_{(0)}\| + \|X\|_\infty)\left(\tfrac{ecT}{n}\right)^n \quad \textit{for all } n > cT.$$

Proof The proof is provided in Appendix A. □

The following general convergence theorem is the main result of Section 6.

[4] Typically, the *pair* (X, Z) would be referred to as the *solution* of the BSDE (6.4). For simplicity of notation, and since Z is not required for our further analysis, here and in the following we also refer to X alone as the *solution* of (6.4).

Theorem 6.3 (Fixed Point and Convergence) *Let $t_0 \in [0,T]$ and $y_0 \in \mathbb{R}$. Then there is a unique progressively measurable process $X^{t_0,y_0} \in \mathcal{D}_\infty$ that solves* (6.2)*. Moreover there are constants $0 < \underline{h} < \overline{h}$ such that $\underline{h} \le X^{t_0,y_0} \le \overline{h}$ for all $(t_0,y_0) \in [0,T] \times \mathbb{R}$. The sequence defined by $X^{t_0,y_0}_{(0)} \triangleq \hat{\varepsilon}$ and $X^{t_0,y_0}_{(n)} \triangleq \varPsi X^{t_0,y_0}_{(n-1)}$ $(n = 1,2,\dots)$ satisfies*

$$\|X^{t_0,y_0}_{(n)} - X^{t_0,y_0}\|_\infty \le C\left(\tfrac{c}{n}\right)^n \text{ for all } n > \tfrac{c}{e}$$

where the constants $C, c > 0$ are explicitly given by $C \triangleq e^{T\|r\|_\infty}(\hat{\varepsilon} + \overline{h})$ and

$$c \triangleq eTq|\tfrac{\delta^\psi}{1-q}|\underline{h}^{q-1} \text{ for } q < 1, \quad c \triangleq eTq|\tfrac{\delta^\psi}{1-q}|\overline{h}^{q-1} \text{ for } q > 1. \qquad (6.5)$$

In order to prove Theorem 6.3 we distinguish the cases $q < 1$ and $q > 1$.

Proof of Theorem 6.3 for $q < 1$. Throughout this paragraph we assume that $q < 1$. We consider the operator $\varPsi$ defined in (6.3) on the closed subset

$$A^{<1} \triangleq \{X \in \mathcal{D}_\infty \,:\, X_t \ge \underline{h},\ \mathrm{d}\,t \otimes P\text{-a.e.}\}, \quad \text{where } \underline{h} \triangleq \hat{\varepsilon}e^{-T\|\tilde{r}\|_\infty} > 0, \quad (6.6)$$

of the Banach space $\mathcal{D}_\infty$. Note that $A^{<1} \subset \mathcal{D}_b$.

Lemma 6.4 *The operator $\varPsi : A^{<1} \to A^{<1}$ is well-defined, and with $c \triangleq |\frac{\delta^\psi}{1-q}|q\underline{h}^{q-1}$ we have*

$$|(\varPsi X)_t - (\varPsi \tilde{X})_t| \le c\textstyle\int_t^T e^{(s-t)\|\tilde{r}\|_\infty}\, \mathrm{E}_t[|X_s - \tilde{X}_s|]\,\mathrm{d}\,s \quad \text{for all } X, \tilde{X} \in A^{<1}.$$

Proof For $X \in A^{<1}$ we obviously have $0 \vee X = X$ and thus

$$\begin{aligned}(\varPsi X)_t &= \mathrm{E}_t\left[\textstyle\int_t^T e^{-\int_t^s \tilde{r}(\eta_\tau)\,\mathrm{d}\,\tau}\frac{\delta^\psi}{1-q}X_s^q\,\mathrm{d}\,s + \hat{\varepsilon}e^{-\int_t^T \tilde{r}(\eta_\tau)\,\mathrm{d}\,\tau}\right]\\ &\ge \mathrm{E}_t\left[\hat{\varepsilon}e^{-\int_t^T \tilde{r}(\eta_\tau)\,\mathrm{d}\,\tau}\right] \ge \underline{h}.\end{aligned}$$

Moreover, $(\varPsi X)_t \le Te^{T\|\tilde{r}\|_\infty}\frac{\delta^\psi}{1-q}(\underline{h}^q + \|X\|^q_\infty) + \hat{\varepsilon}e^{T\|\tilde{r}\|_\infty}$ and it follows that $\varPsi : A^{<1} \to A^{<1}$ is well-defined. For the second part of the claim, note that the mapping $[\underline{h},\infty) \to \mathbb{R}, x \mapsto x^q$ is Lipschitz continuous with Lipschitz constant $L \triangleq q\underline{h}^{q-1}$. Thus we have

$$|(\varPsi X)_t - (\varPsi \tilde{X})_t| \le \tfrac{\delta^\psi}{1-q}L\textstyle\int_t^T e^{(s-t)\|\tilde{r}\|_\infty}\, \mathrm{E}_t[|X_s - \tilde{X}_s|]\,\mathrm{d}\,s. \qquad \square$$

Theorem 6.5 (Fixed Point and Convergence: $q < 1$) *Suppose that $q < 1$. There exists a progressively measurable process $X \in \mathcal{D}_\infty$ with*

$$X_t = (\varPsi X)_t = \mathrm{E}_t\left[\textstyle\int_t^T e^{-\int_t^s \tilde{r}(\eta_\tau)\,\mathrm{d}\,\tau}\frac{\delta^\psi}{1-q}X_s^q\,\mathrm{d}\,s + \hat{\varepsilon}e^{-\int_t^T \tilde{r}(\eta_\tau)\,\mathrm{d}\,\tau}\right].$$

Starting from $X^{(0)} = \hat{\varepsilon}$ the iterative sequence $\underline{h} \le X^{(n)} \triangleq \varPsi X^{(n-1)}$ $(n = 1,2,\dots)$ converges to X in $\mathcal{D}_\infty$. In addition we have $\underline{h} \le X \le \overline{h}$ where both $\underline{h}$, given by (6.6)*, and $\overline{h} > 0$ are independent of (t_0,y_0). Finally, the process X is the unique fixed point of $\varPsi$ that is bounded below by $\underline{h}$.*

Proof It is clear that $\underline{h} \le \hat\varepsilon = X^{(0)}$ and thus $X^{(0)} \in A^{<1}$. Lemma 6.4 implies that $X^{(n)} \in A^{<1}$ for each member of the sequence $X^{(n)} = \Psi X^{(n-1)}$. Applying Proposition 6.2 to the mapping $\Psi : A^{<1} \to A^{<1}$ it follows that this sequence converges in norm to the unique fixed point $X = \Psi X$. In particular we have $0 < \underline{h} \le X$ and

$$X_t = (\Psi X)_t = \mathrm{E}_t\big[\int_t^T e^{-\int_t^s \tilde r(\eta_\tau)\,\mathrm{d}\tau} \tfrac{\delta^\psi}{1-q} X_s^q \,\mathrm{d}\, s + \hat\varepsilon e^{-\int_t^T \tilde r(\eta_\tau)\,\mathrm{d}\tau}\big].$$

To establish the upper bound, observe that by Lemma 6.1, X satisfies

$$\mathrm{d}\, X_t = -\left[\tfrac{\delta^\psi}{1-q} X_t^q - \tilde r(t,\eta_t) X_t\right]\mathrm{d}\, t + \mathrm{d}\, M_t, \quad X_T = \hat\varepsilon$$

where M is an L^2-martingale. Hence for every stopping time τ we have

$$\begin{aligned} 1_{\{\tau>t\}} X_t &= \mathrm{E}_t\left[1_{\{\tau>t\}} \int_t^\tau \left(\tfrac{\delta^\psi}{1-q} X_s^q - \tilde r(s,\eta_s) X_s\right)\mathrm{d}\, s + 1_{\{\tau>t\}} X_\tau\right] \\ &\le \mathrm{E}_t\left[1_{\{\tau>t\}} \int_t^\tau (aX_s + b)\,\mathrm{d}\, s + 1_{\{\tau>t\}} X_\tau\right] \end{aligned}$$

where $a \triangleq \frac{\delta^\psi}{1-q} + \|\tilde r\|_\infty > 0$ and $b \triangleq \frac{\delta^\psi}{1-q}(1+\underline{h}^q)$. Thus we can apply a variant of the stochastic Gronwall-Bellman inequality, see Proposition B.1, to obtain

$$X_t \le \mathrm{E}_t\left[\int_t^T e^{a(s-t)} b \,\mathrm{d}\, s + e^{a(T-t)} \hat\varepsilon\right] \le T e^{aT} b + e^{aT}\hat\varepsilon \triangleq \overline{h}$$

where $\overline{h}$ is a constant depending only on $\delta, \psi, q, \tilde r, \hat\varepsilon$ and T. □

Proof (of Theorem 6.3 for $q < 1$) Theorem 6.5 yields a unique process X^{t_0,y_0} that satisfies $X^{t_0,y_0} = \Psi X^{t_0,y_0}$ and $0 < \underline{h} \le X^{t_0,y_0} \le \overline{h} < \infty$, where the constants $\underline{h}, \overline{h}$ are independent of (t_0, y_0). By Lemma 6.1 the process X^{t_0,y_0} is a solution of (6.2). Proposition 6.2 shows that the convergence rate of the iterative sequence $X^{t_0,y_0}_{(0)} = \hat\varepsilon$, $X^{t_0,y_0}_{(n)} \triangleq \Psi X^{t_0,y_0}_{(n-1)}$ $(n = 1, 2, \dots)$ is given by

$$\|X^{t_0,y_0}_{(n)} - X^{t_0,y_0}\|_\infty \le e^{T\varrho}(\|X^{t_0,y_0}_{(0)}\| + \|X^{t_0,y_0}\|_\infty)\left(\tfrac{ecT}{n}\right)^n$$

where $\varrho \triangleq \|\tilde r\|_\infty$ and $c \triangleq |\frac{\delta^\psi}{1-q}| q \underline{h}^{q-1}$ by Lemma 6.4. In view of the fact that $\|X^{t_0,y_0}_{(0)}\|_\infty = \hat\varepsilon$ and $\|X^{t_0,y_0}\|_\infty \le \overline{h}$, this completes the proof. □

Proof of Theorem 6.3 for $q > 1$. In the following we suppose that $q > 1$. In contrast to the previous paragraph, we now consider the operator Ψ on

$$A^{>1} \triangleq \{X \in \mathcal{D}_\infty : X_t \le \overline{h},\ \mathrm{d}\, t \otimes P\text{-a.e.}\}, \quad \text{where } \overline{h} \triangleq \hat\varepsilon e^{T\|\tilde r\|_\infty}. \tag{6.7}$$

Note that $A^{>1} \subset \mathcal{D}_\infty$ is closed and satisfies $A^{>1} \subset \mathcal{D}_b$.

Lemma 6.6 *The operator $\Psi : A^{>1} \to A^{>1}$ is well-defined, and with $c \triangleq |\frac{\delta^\psi}{1-q}| q \overline{h}^{q-1}$ we have*

$$|(\Psi X)_t - (\Psi \tilde X)_t| \le c \int_t^T e^{(s-t)\|\tilde r\|_\infty}\, \mathrm{E}_t[|X_s - \tilde X_s|]\,\mathrm{d}\, s \quad \textit{for all } X, \tilde X \in A^{>1}.$$

Proof For any $X \in \mathcal{D}_\infty$ we have

$$(\Psi X)_t = \mathrm{E}_t\left[\int_t^T e^{-\int_t^s \tilde{r}(\eta_\tau)\,\mathrm{d}\,\tau} \tfrac{\delta^\psi}{1-q}(0 \vee X_s)^q \,\mathrm{d}\,s + \hat{\varepsilon} e^{-\int_t^T \tilde{r}(\eta_\tau)\,\mathrm{d}\,\tau}\right]$$
$$\leq \mathrm{E}_t\left[\hat{\varepsilon} e^{-\int_t^T \tilde{r}(\eta_\tau)\,\mathrm{d}\,\tau}\right] \leq \overline{h}.$$

In addition we have $(\Psi X)_t \geq -Te^{T\|\tilde{r}\|_\infty}|\frac{\delta^\psi}{1-q}|\|X\|_\infty^q$ so that $\Psi : A^{>1} \to A^{>1}$ is well-defined. Since the function $[0, \overline{h}] \to \mathbb{R}, x \mapsto (0 \vee x)^q$ is Lipschitz with Lipschitz constant $L \triangleq q\overline{h}^{q-1}$ we obtain

$$|(\Psi X - \Psi \tilde{X})(t, y)| \leq |\tfrac{\delta^\psi}{1-q}| L \int_t^T e^{(s-t)\|\tilde{r}\|_\infty} \mathrm{E}_t[|X_s - \tilde{X}_s|] \,\mathrm{d}\,s. \qquad \square$$

Theorem 6.7 (Fixed Point and Convergence: $q > 1$**)** *Let $q > 1$. There is a progressively measurable process $X \in \mathcal{D}_\infty$ with*

$$X_t = (\Psi X)_t = \mathrm{E}_t\big[\int_t^T e^{-\int_t^s \tilde{r}(\eta_\tau)\,\mathrm{d}\,\tau} \tfrac{\delta^\psi}{1-q} X_s^q \,\mathrm{d}\,s + \hat{\varepsilon} e^{-\int_t^T \tilde{r}(\eta_\tau)\,\mathrm{d}\,\tau}\big]$$

and the iterative sequence $\overline{h} \geq X^{(n)} \triangleq \Psi X^{(n-1)}$ $(n = 1, 2, \dots)$ with $X^{(0)} \triangleq \hat{\varepsilon}$ converges to X in $\mathcal{D}_\infty$. Besides we have $\underline{h} \leq X \leq \overline{h}$ where $\underline{h} > 0$ and $\overline{h}$ in (6.7) *are independent of (t_0, y_0), and X is the unique positive fixed point of Ψ that is bounded above by $\overline{h}$.*

Proof We have $X^{(0)} = \hat{\varepsilon} \leq \overline{h}$ and thus $X^{(0)} \in A^{>1}$. By Lemma 6.6 each member of the iterative sequence satisfies $X^{(n)} = \Psi X^{(n-1)} \in A^{>1}$ and in particular $X^{(n)} \leq \overline{h}$. Proposition 6.2 applies to $\Psi : A^{>1} \to A^{>1}$ to show that there is a unique $X \in A^{>1}$ with $\Psi X = X$ and $\|X^{(n)} - X\|_\infty \to 0$. In particular X satisfies $X \leq \overline{h}$.

To demonstrate that $X \geq 0$, recall from Lemma 6.1 that

$$\mathrm{d}\,X_t = -\left[\tfrac{\delta^\psi}{1-q}(0 \vee X_t)^q - \tilde{r}(t, \eta_t)X_t\right]\mathrm{d}\,t + \mathrm{d}\,M_t, \quad X_T = \hat{\varepsilon}$$

with an L^2-martingale M. Thus for all stopping times τ we have

$$1_{\{\tau>t\}}X_t = \mathrm{E}_t\left[1_{\{\tau>t\}}\int_t^\tau \left(\tfrac{\delta^\psi}{1-q}(0 \vee X_s)^q - \tilde{r}(s, \eta_s)X_s\right)\mathrm{d}\,s + 1_{\{\tau>t\}}X_\tau\right].$$

With $L \triangleq q\overline{h}^{q-1}$ denoting the Lipschitz constant of $(-\infty, \overline{h}) \to \mathbb{R}, x \mapsto (0 \vee x)^q$, we obtain

$$1_{\{\tau>t\}}X_t \geq \mathrm{E}_t\left[1_{\{\tau>t\}}\int_t^\tau \left(\tfrac{\delta^\psi}{1-q}L1_{\{X_s>0\}}X_s - \tilde{r}(s, \eta_s)X_s\right)\mathrm{d}\,s + 1_{\{\tau>t\}}X_\tau\right]$$
$$= \mathrm{E}_t\left[1_{\{\tau>t\}}\int_t^\tau a_s X_s \,\mathrm{d}\,s + 1_{\{\tau>t\}}X_\tau\right]$$

where the process $a_s \triangleq \frac{\delta^\psi}{1-q}L1_{\{X_s>0\}} - \tilde{r}(s, \eta_s)$ is bounded and progressively measurable. Now Proposition B.1 yields

$$X_t \geq \mathrm{E}_t\left[e^{\int_t^T a_s \,\mathrm{d}\,s}\hat{\varepsilon}\right] \geq e^{T(\frac{\delta^\psi}{1-q}L - \|r\|_\infty)}\hat{\varepsilon} \triangleq \underline{h} > 0$$

where $\underline{h}$ is a constant that depends only on $\delta, \psi, q, \tilde{r}, \hat{\varepsilon}$ and T. In particular X is positive and we have

$$X_t = (\Psi X)_t = \mathrm{E}_t\big[\int_t^T e^{-\int_t^s \tilde{r}(\eta_\tau)\,\mathrm{d}\,\tau} \tfrac{\delta^\psi}{1-q} X_s^q \,\mathrm{d}\,s + \hat{\varepsilon} e^{-\int_t^T \tilde{r}(\eta_\tau)\,\mathrm{d}\,\tau}\big]. \qquad \square$$

Proof (of Theorem 6.3 for $q > 1$) The proof is the same as in the case $q < 1$, with Theorem 6.5 replaced by Theorem 6.7 and Lemma 6.4 replaced by Lemma 6.6. □

6.2 Differentiability of the Fixed Point

In this section we demonstrate that the solutions X^{t_0,y_0} of (6.2) provided by Theorem 6.3 yield a solution h to the reduced HJB equation (4.5)

$$h_t - \tilde{r}h + \tilde{\alpha}h_y + \tfrac{1}{2}\beta^2 h_{yy} + \tfrac{\delta^\psi}{1-q} h^q = 0, \quad h(T,\,\cdot\,) = \hat{\varepsilon}.$$

For that purpose we cut off the nonlinearity using the a priori estimates provided by Theorem 6.3, which leads us to a PDE that is known to have a classical solution $g \in C_b^{1,2}([0,T]\times\mathbb{R})$. We then conclude by proving that $g = h$, where $h(t_0,y_0) = X_{t_0}^{t_0,y_0}$. Here and in the following $C_b^{1,2}([0,T]\times\mathbb{R})$ denotes the Banach space of all functions $u : [0,T]\times\mathbb{R}, (t,y) \to u(t,y)$ that are once continuously differentiable with respect to t and twice continuously differentiable with respect to y and have norm $\|u\|_{C^{1,2}} < \infty$. The norm is given by

$$\|u\|_{C^{1,2}} \triangleq \|u\|_\infty + \|u_t\|_\infty + \|u_y\|_\infty + \|u_{yy}\|_\infty \quad \text{for } u \in C_b^{1,2}([0,T]\times\mathbb{R}).$$

Theorem 6.8 (Differentiability, Probabilistic Representation)
Let X^{t_0,y_0} denote the solutions to the FBSDEs (6.2) *given by Theorem 6.3 and define*

$$h(t_0,y_0) \triangleq X_{t_0}^{t_0,y_0} \quad \textit{for } (t_0,y_0) \in [0,T]\times\mathbb{R}.$$

Then $h \in C_b^{1,2}([0,T]\times\mathbb{R})$ and h satisfies the reduced HJB equation (4.5). *Moreover, h is the unique solution of* (4.5) *that is bounded from above and away from* 0. *In addition, h admits the probabilistic representation*

$$h(t,y) = \mathrm{E}_t\left[\int_t^T \left(-\tilde{r}(\eta_s^{t,y})h(s,\eta_s^{t,y}) + \tfrac{\delta^\psi}{1-q} h(s,\eta_s^{t,y})^q\right)\mathrm{d}\,s + \hat{\varepsilon}\right]. \qquad (6.8)$$

Proof We take $\underline{h}$ and $\overline{h}$ as in Theorem 6.3 and choose a smooth cut-off function $\varphi \in C_b^1(\mathbb{R})$ with

$$\varphi(v) = \tfrac{1}{2}\underline{h} \text{ for } v \le \tfrac{1}{2}\underline{h}, \quad \varphi(v) = v \text{ for } v \in [\underline{h},\overline{h}], \quad \varphi(v) = \overline{h}+1 \text{ for } v \ge \overline{h}+1.$$

We set $f(v) \triangleq \frac{\delta^\psi}{1-q}\varphi(v)^q$ and consider the semilinear Cauchy problem

$$g_t - \tilde{r}g + \tilde{\alpha}g_y + \tfrac{1}{2}\beta^2 g_{yy} + f(g) = 0, \quad g(T,\,\cdot\,) = \hat{\varepsilon}. \qquad (6.9)$$

The function f is clearly continuously differentiable and bounded with a bounded derivative. Hence by a classical result on semilinear PDEs, see Corollary C.4, there exists a unique classical solution $g \in C_b^{1,2}([0,T] \times \mathbb{R})$ to (6.9).

To show that $g = h$ we fix $(t_0, y_0) \in [0,T] \times \mathbb{R}$ and set $\bar{X}_t^{t_0,y_0} \triangleq \bar{X}_t \triangleq g(t, \eta_t)$, $t \in [t_0, T]$, where $\eta \triangleq \eta^{t_0,y_0}$ is given by (6.1). By Itō's formula and (6.9)

$$\mathrm{d}\,\bar{X}_t = -[f(\bar{X}_t) - \tilde{r}(\eta_t)\bar{X}_t]\,\mathrm{d}\,t + \bar{Z}_t\,\mathrm{d}\,W_t, \quad \bar{X}_T = \hat{\varepsilon}, \tag{6.10}$$

where $\bar{Z}_t \triangleq g_y(t, \eta_t)\beta(\eta_t^{u,y})$ is bounded. On the other hand, Theorem 6.3 yields a unique solution $X \triangleq X^{t_0,y_0}$ of (6.2), i.e.

$$\mathrm{d}\,X_t = -\left[\tfrac{\delta^\psi}{1-q} X_t^q - \tilde{r}(\eta_t)X_t\right]\mathrm{d}\,t + Z_t\,\mathrm{d}\,W_t, \quad X_T = \hat{\varepsilon}.$$

Since $\underline{h} \leq X \leq \overline{h}$ we have $f(X_t) = \frac{\delta^\psi}{1-q} X_t^q$ and therefore X also satisfies

$$\mathrm{d}\,X_t = -[f(X_t) - \tilde{r}(\eta_t)X_t]\,\mathrm{d}\,t + Z_t\,\mathrm{d}\,W_t, \qquad X_T = \hat{\varepsilon}.$$

Thus we conclude that X solves (6.10), too. Since (6.10) is a BSDE with a Lipschitz driver, it follows from Theorem 2.1 in El Karoui, Peng and Quenez [20] that $X = \bar{X}$. In particular we have $h(t_0, y_0) = X_{t_0}^{t_0,y_0} = \bar{X}_{t_0}^{t_0,y_0} = g(t_0, y_0)$.

To show uniqueness, let $0 < \underline{u} \leq u \leq \overline{u}$ satisfy (4.5). Replacing $\underline{h}$ and $\overline{h}$ by $\underline{h} \wedge \underline{u}$ and $\overline{h} \vee \overline{u}$ in the first part of the proof, it follows that both u and h coincide with the unique solution of (6.9). □

Proof (of Theorem 4.6) This follows from Theorem 6.3 and Theorem 6.8. □

7 PDE Iteration Approach

In this section we develop an explicit constructive method to obtain the solution of the reduced HJB equation. Existence and uniqueness of the solution are guaranteed by Theorem 4.6 above. More precisely, we will show that $h_n \triangleq \Phi^n \hat{\varepsilon} \overset{n\to\infty}{\longrightarrow} h$ in $C^{0,1}$, where the operator Φ is given by

$$\Phi : D(\Phi) \subset C_b^{1,2}([0,T] \times \mathbb{R}) \to C_b^{1,2}([0,T] \times \mathbb{R}), \quad f \mapsto \Phi f$$

and $g \triangleq \Phi f$ is the unique classical solution of the linear Cauchy problem

$$0 = g_t - \tilde{r} g + \tilde{\alpha} g_y + \tfrac{1}{2}\beta^2 g_{yy} + \tfrac{\delta^\psi}{1-q}(0 \vee f)^q \quad \text{with} \quad g(T, \cdot\,) = \hat{\varepsilon}.$$

Thus h can be determined by iteratively solving linear PDEs.

7.1 PDE Iteration

Our first step is to show that the iteration of PDEs as above is feasible. Thus we verify that the operator Φ is well-defined on its domain $D(\Phi)$ where

$$D(\Phi) \triangleq \{f \in C_b^{1,2}([0,T]\times\mathbb{R}) \,:\, f \geq \underline{h}\} \quad \text{for } q<1, \quad \text{and}$$
$$D(\Phi) \triangleq \{f \in C_b^{1,2}([0,T]\times\mathbb{R}) \,:\, f \leq \overline{h}\} \quad \text{for } q>1.$$

Here $\underline{h}, \overline{h}$ are the constants specified in Theorem 6.3.

Lemma 7.1 *If $u \in D(\Phi)$ then there exists a unique $g \in C^{1,2}([0,T]\times\mathbb{R})$ with*

$$0 = g_t - \tilde{r}g + \tilde{\alpha}g_y + \tfrac{1}{2}\beta^2 g_{yy} + \tfrac{\delta^\psi}{1-q}(0 \vee u)^q, \quad g(T,\cdot\,) = \hat{\varepsilon}. \tag{7.1}$$

Proof If $q<1$ and $u \geq \underline{h} > 0$, then $f \triangleq \frac{\delta^\psi}{1-q}(0\vee u)^q \in C^{1,2}([0,T]\times\mathbb{R})$. If $q>1$ with $u \leq \overline{h} < \infty$, then f is Lipschitz continuous since

$$|f(t,y) - f(t',y')| \leq |\tfrac{\delta^\psi}{1-q}|q\overline{h}^{q-1}|u(t,y)-u(t',y')|.$$

In either case Corollary C.2 yields a unique $g \in C^{1,2}([0,T]\times\mathbb{R})$ satisfying (7.1). □

To establish the link between the iterated solutions h_n of the Cauchy problem and the stochastic processes $X_{(n)}^{t_0,y_0}$ of Section 6, we first record a simple uniqueness result:

Lemma 7.2 *For every $n \in \mathbb{N}$ the process $X^{(n)} \triangleq X_{(n)}^{t_0,y_0}$ defined in Theorem 6.3 is the unique solution of the linear BSDE*

$$\mathrm{d}\,X_t^{(n)} = -\Big[\tfrac{\delta^\psi}{1-q}\big(0\vee X_t^{(n-1)}\big)^q - \tilde{r}(\eta_t^{t_0,y_0})X_t^{(n)}\Big]\,\mathrm{d}\,t + Z_t^{(n)}\,\mathrm{d}\,W_t, \quad X_t^{(n)} = \hat{\varepsilon}. \tag{7.2}$$

Proof With $\varphi \triangleq \frac{\delta^\psi}{1-q}(0\vee X^{(n-1)})^q$, by definition of $X^{(n)}$, we have

$$X_t^{(n)} = \mathrm{E}_t\left[\int_t^T e^{-\int_t^s \tilde{r}(\eta_\tau^{t_0,y_0})\,\mathrm{d}\,\tau}\varphi_s\,\mathrm{d}\,s + \hat{\varepsilon}e^{-\int_t^T \tilde{r}(\eta_\tau^{t_0,y_0})\,\mathrm{d}\,\tau}\right].$$

By Proposition 2.2 in [20], $X^{(n)}$ is the unique solution of the linear backward equation $\mathrm{d}\,X_t^{(n)} = -[\varphi_t - \tilde{r}(\eta_t^{t_0,y_0})X_t^{(n)}]\,\mathrm{d}\,t + Z_t^{(n)}\,\mathrm{d}\,W_t$. □

The connection between h_n and $X_{(n)}^{t_0,y_0}$ is now given as follows:

Theorem 7.3 *For each $n\in\mathbb{N}$ we have $h_n = \Phi^n\hat{\varepsilon} \in D(\Phi)$ and*

$$h_n(t,\eta_t^{t_0,y_0}) = \big(X_{(n)}^{t_0,y_0}\big)_t \quad \textit{for all } t\in[t_0,T],\ (t_0,y_0)\in[0,T]\times\mathbb{R}.$$

Proof The assertion is clearly true for $n = 0$ since $h_0 = \Phi^0 \hat{\varepsilon} = \hat{\varepsilon}$ and $X_{(0)}^{t_0,y_0} = \hat{\varepsilon}$. Assume by induction that $h_{n-1} = \Phi^{n-1}\hat{\varepsilon} \in D(\Phi)$ with

$$h_{n-1}(t, \eta_t^{t_0,y_0}) = \big(X_{(n-1)}^{t_0,y_0}\big)_t \quad \text{for all } t \in [t_0, T],\ (t_0, y_0) \in [0,T] \times \mathbb{R}. \tag{7.3}$$

By Lemma 7.1 $g \triangleq h_n = \Phi h_{n-1} \in C_b^{1,2}([0,T] \times \mathbb{R})$ is well-defined and satisfies

$$0 = g_t - \tilde{r} g + \tilde{\alpha} g_y + \tfrac{1}{2}\beta^2 g_{yy} + \tfrac{\delta^\psi}{1-q}(0 \vee h_{n-1})^q, \quad g(T, \cdot\,) = \hat{\varepsilon}. \tag{7.4}$$

Let $(t_0, y_0) \in [0,T] \times \mathbb{R}$ and $\eta \triangleq \eta^{t_0,y_0}$ be given by (6.1) and set $X_t \triangleq g(t, \eta_t)$. By (7.3), (7.4) and Itō's formula we have

$$\mathrm{d}\, X_t = -\Big[\tfrac{\delta^\psi}{1-q}\Big(0 \vee \big(X_{(n-1)}^{t_0,y_0}\big)_t\Big)^q - \tilde{r}(\eta_t) X_t\Big]\, \mathrm{d}\, t + Z_t\, \mathrm{d}\, W_t,$$

where $Z_t \triangleq g_y(t, \eta_t)\beta(\eta_t)$ is bounded. Consequently X is a solution of (7.2), so by Lemma 7.2 we must have $X = X_{(n)}^{t_0,y_0}$. Hence it follows that

$$h_n(t, \eta_t^{t_0,y_0}) = \big(X_{(n)}^{t_0,y_0}\big)_t \quad \text{for all } t \in [t_0, T],\ (t_0, y_0) \in [0,T] \times \mathbb{R}.$$

For $q < 1$ Theorem 6.5 implies $\underline{h} \le X_{(n)}^{t_0,y_0}$, whereas for $q > 1$ we have $X_{(n)}^{t_0,y_0} \le \overline{h}$ by Theorem 6.7. Thus $h_n \in D(\Phi)$, and the induction is complete. □

The convergence $h_n \to h$ is now a corollary of the analysis in Section 6.

Corollary 7.4 *Let $h \in C_b^{1,2}([0,T] \times \mathbb{R})$ be the unique solution to the reduced HJB equation* (4.5)*. Moreover let $h_n \triangleq \Phi^n \hat{\varepsilon} \in C_b^{1,2}([0,T] \times \mathbb{R})$ be defined recursively as the unique bounded solution of the Cauchy problem*

$$0 = (h_n)_t - \tilde{r} h_n + \tilde{\alpha}(h_n)_y + \tfrac{1}{2}\beta^2 (h_n)_{yy} + \tfrac{\delta^\psi}{1-q}(0 \vee h_{n-1})^q, \quad h_n(T, \cdot\,) = \hat{\varepsilon}.$$

Then, with the constants $C, c > 0$ given by (6.5)*, we have*

$$\|h_n - h\|_\infty \le C\left(\tfrac{c}{n}\right)^n \quad \text{for all } n > \tfrac{c}{e}.$$

Proof By Theorem 7.3 we have $h_n(t, \eta_t^{t_0,y_0}) = \big(X_{(n)}^{t_0,y_0}\big)_t$ for all $t \in [t_0, T]$ and all $(t_0, y_0) \in [0,T] \times \mathbb{R}$. Thus Theorem 6.3 yields

$$|h_n(t_0, y_0) - h(t_0, y_0)| = |\big(X_{(n)}^{t_0,y_0}\big)_{t_0} - X_{t_0}^{t_0,y_0}| \le \|X_{(n)}^{t_0,y_0} - X^{t_0,y_0}\|_\infty \le C\left(\tfrac{c}{n}\right)^n$$

for all $n > \frac{c}{e}$ uniformly in $(t_0, y_0) \in [0,T] \times \mathbb{R}$. □

7.2 Convergence Rate of the PDE Iteration in $C^{0,1}$

In this section we use the probabilistic representation (6.8) of h established in Theorem 6.8 to demonstrate that both h_n and $(h_n)_y$ converge uniformly to h and h_y. We also identify the relevant convergence rate. We replace (A1) by the slightly stronger regularity condition

(A1') The coefficients $r, \lambda, \sigma, \alpha, \beta$ are bounded with bounded, Lipschitz continuous derivatives.

Similarly as in Lemma 4.5 this assumption guarantees that $\tilde{\alpha}$ and β have a bounded Lipschitz continuous derivative. This implies the following estimate for the derivative of the semigroup $(P_s)_{s\in[0,T]}$ generated by $\eta^{0,\,\cdot}$:

Proposition 7.5 (Derivative of the Semigroup) *Assume that (A1') and (A2) are satisfied and let* $(P_s)_{s\in[0,T]}$ *be the semigroup associated with the process* $\eta^{0,\,\cdot}$ *given by* (6.1). *Then there exists a constant* $M > 0$ *such that for all* $f \in C_b(\mathbb{R})$ *we have*

$$\|D(P_t f)\|_\infty \leq M t^{-\frac{1}{2}} \|f\|_\infty \text{ for all } t \in [0,T].$$

Proof See Theorem 1.5.2 in Cerrai [11] or Theorem 3.3 in Bertoldi and Lorenzi [4]. □

Remark. We refer to Elworthy and Li [21] and Cerrai [10] for related results. For Hölder-continuous $f \in C_b(\mathbb{R})$, results like Proposition 7.5 are well-known in the literature on parabolic PDEs; see, e.g., [29]. ◇

We are now in a position to establish the convergence of our fixed point iteration in $C^{0,1}([0,T]\times\mathbb{R})$ endowed with the norm $\|h\|_{C^{0,1}} \triangleq \|h\|_\infty + \|\frac{\partial}{\partial y} h\|_\infty$. This provides the rigorous basis for the numerical method we develop in Section 8.

Theorem 7.6 (Convergence in $C^{0,1}$) *The functions* h_n $(n = 1, 2, \dots)$ *are uniformly bounded in* $C^{0,1}([0,T]\times\mathbb{R})$ *and we have*

$$\|h_n - h\|_{C^{0,1}} \leq 2cM\sqrt{T}\left(\|r\|_\infty \tfrac{C}{n} + \tfrac{1}{eT}\right)\left(\tfrac{c}{n-1}\right)^{n-1} \quad \text{for all } n > \tfrac{c}{e} + 1,$$

where $C, c > 0$ *are given by* (6.5) *and* $M > 0$ *is given in Proposition 7.5.*

Proof First, for each $n \in \mathbb{N}$ Theorem 7.3 implies that $h_n(t, \eta_t^{t_0,y_0}) = \left(X_{(n)}^{t_0,y_0}\right)_t$ so by Lemma 7.2

$$h_n(t, \eta_t^{t_0,y_0}) = \mathrm{E}_t\left[\int_t^T \tfrac{\delta^\psi}{1-q}(0 \vee h_{n-1}(s, \eta_s^{t_0,y_0}))^q - \tilde{r}(\eta_s) h_n(s, \eta_s^{t_0,y_0})\,\mathrm{d}\,s + \hat{\varepsilon}\right]$$

for all $t \in [t_0, T]$. Hence with $f_n \triangleq \frac{\delta^\psi}{1-q}(0 \vee h_n)^q$ we can represent h_n via

$$h_n(t_0, y_0) = \int_0^{T-t_0} (P_s \tilde{h}_n(t_0, s, \cdot\,))(y_0)\,\mathrm{d}\,s + \hat{\varepsilon}$$

where $(P_s)_{s\in[0,T]}$ denotes the semigroup corresponding to $\eta^{0,\,\cdot}$ and $\tilde{h}_n(t,s,y) \triangleq f_{n-1}(s+t,y) - \tilde{r}(y)h_n(s+t,y)$. Analogously, by Theorem 6.8 we obtain

$$h(t_0,y_0) = \int_0^{T-t}(P_s\tilde{h}(t_0,s,\,\cdot\,))(y)\,\mathrm{d}\,s + \hat{\varepsilon},$$

with $\tilde{h}(t,s,\cdot) \triangleq \frac{\delta^\psi}{1-q}h(s+t,\cdot)^q - \tilde{r}h(s+t,\cdot)$. Setting $v_n \triangleq \tilde{h}_n - \tilde{h}$, we thus have $h_n(t_0,\,\cdot\,) - h(t_0,\,\cdot\,) = \int_0^{T-t_0} P_s v_n(t_0,s,\,\cdot\,)\,\mathrm{d}\,s$. With $C, c > 0$ given by (6.5) and Corollary 7.4 it therefore follows that

$$\begin{aligned}\|v_n\|_\infty &\le \|r\|_\infty\|h_n - h\|_\infty + L_q|\tfrac{\delta^\psi}{1-q}|\|h_{n-1} - h\|_\infty \\ &\le C\|r\|_\infty(\tfrac{c}{n})^n + L_q|\tfrac{\delta^\psi}{1-q}|(\tfrac{c}{n-1})^{n-1},\end{aligned}$$

where $L_q \triangleq 1_{\{q<1\}}q|\frac{\delta^\psi}{1-q}|\underline{h}^{q-1} + 1_{\{q>1\}}q|\frac{\delta^\psi}{1-q}|\overline{h}^{q-1}$. Proposition 7.5 implies

$$\|\tfrac{\partial}{\partial y}h_n(t_0,\,\cdot\,) - \tfrac{\partial}{\partial y}h(t_0,\,\cdot\,)\|_\infty \le M\|v_n\|_\infty\int_0^{T-t_0}\tfrac{1}{\sqrt{s}}\,\mathrm{d}\,s \le 2\sqrt{T}M\|v_n\|_\infty.$$

Since $L_q|\frac{\delta^\psi}{1-q}| = \frac{c}{eT}$ and $(\frac{c}{n})^{n-1} \le (\frac{c}{n-1})^{n-1}$ this completes the proof. □

8 Numerical Results

8.1 User's Guide

Before we study specific applications, we provide a general outline that explains how to apply our theoretical results to concrete consumption-portfolio problems. By Theorem 5.1, the solution to the consumption-portfolio problem (P) is given by the optimal policies $(\hat{\pi},\hat{c})$ in (5.1). These depend on the solution of the reduced HJB equation

$$0 = h_t - \tilde{r}h + \tilde{\alpha}h_y + \tfrac{1}{2}\beta^2 h_{yy} + \tfrac{\delta^\psi}{1-q}h^q, \quad h(T,\,\cdot\,) = \hat{\varepsilon}, \tag{4.5}$$

see also Definition 4.3. Theorem 4.6 implies that this PDE admits a unique bounded classical solution. Algorithm 8.1 below provides a step-by-step method for the construction of solutions to PDEs of the form (4.5). This algorithm is easy to implement and relies solely on an efficient method for solving linear PDEs as a prerequisite. Consistency of this approach is guaranteed by Theorem 7.6, which demonstrates that the sequence of solutions provided by Algorithm 8.1 converges to the solution of (4.5). Theorem 7.6 also implies that the same is true for the associated derivatives. Additionally, Theorem 7.6 ensures a superexponential speed of convergence.

Algorithm 8.1

(1) Set $h_0 \triangleq \hat{\varepsilon}$ and $n \triangleq 1$.

(2) Compute h_n as the solution g of the linear inhomogeneous PDE

$$0 = g_t - \tilde{r}g + \tilde{\alpha}g_y + \tfrac{1}{2}\beta^2 g_{yy} + \tfrac{\delta^\psi}{1-q}(0 \vee h_{n-1})^q, \quad g(T,\,\cdot\,) = \hat{\varepsilon}. \qquad (*)$$

(3) If h_n is not sufficiently close to h_{n-1}, increase n by 1 and return to (2).

To solve the linear PDE $(*)$ in Step (2), we use a semi-implicit Crank-Nicolson scheme. Notice that the relevant finite-difference matrices depend on the linear part of the PDE $(*)$ only. Therefore, the construction and LU decomposition of these matrices must be carried out only once in a precomputation step. This is one important feature that contributes to the excellent numerical performance of our method.

Remark. Our results require the coefficients of the state process to satisfy assumptions (A1') and (A2). These are standard regularity conditions, but they may not be satisfied for specific models such as the Heston model. In this case, our analysis applies to a modification of the model defined in terms of a suitable approximation of the identity function (see below). ◇

8.2 Generalized Square-Root and GARCH Diffusions

We first illustrate our approach for the model specification

$$\mathrm{d}\,S_t = S_t[(r + \bar{\lambda}I(Y_t))\,\mathrm{d}\,t + \sqrt{I_+(Y_t)}\,\mathrm{d}\,W_t] \qquad (8.1)$$

with constant interest rate r and constant $\bar{\lambda}$, i.e. we consider a stochastic volatility model with stochastic excess return. The state process satisfies

$$\mathrm{d}\,Y_t = (\vartheta - \kappa I(Y_t))\,\mathrm{d}\,t + \bar{\beta}I_+(Y_t)^p(\rho\,\mathrm{d}\,W_t + \sqrt{1-\rho^2}\,\mathrm{d}\,\bar{W}_t) \qquad (8.2)$$

with mean reversion level ϑ/κ, mean reversion speed κ, and $p \in [0.5, 1]$. Here I and I_+ are smooth approximations of the identity function on $\mathbb{R}$ and $\mathbb{R}_+$, in the following sense: $I, I_+ \in C_b^\infty(\mathbb{R})$ with bounded derivatives of all orders,

$$I(y) = y \text{ for all } y \in [-K, K], \quad I_+(y) = y \text{ for } y \in [\tfrac{1}{K}, K]$$

and $I_+(y) \geq 1/(2K)$ for all $y \in \mathbb{R}$ where $K > 0$ is a (numerically large, but fixed) constant. This model satisfies (A1') and (A2). For $p = 0.5$ we obtain a truncated Heston [23] model and for $p = 1$ a truncated GARCH diffusion model. Christoffersen, Jacobs and Mimouni [13] test the empirical performance of stochastic volatility models and find that models with $p = 1$ outperform the Heston model. Note that closed-form solutions for consumption-portfolio problems with such dynamics are only available in the special case $p = 0.5$, but solely with specific parameter choices. Further note that for $p > 0.5$ the model is not affine, i.e. explicit solutions cannot be expected. The model coefficients are chosen as follows:

$$r = 0.02,\ \kappa = 5,\ \tfrac{\vartheta}{\kappa} = 0.15^2,\ \bar{\lambda} = 3.11,\ \rho = -0.5,\ \text{and } \bar{\beta} = 0.25 \qquad (8.3)$$

so that for $p = 0.5$ the calibration is similar to that of Liu and Pan [31]. Furthermore, we assume that the agent's rate of time preference is $\delta = 0.05$ and that the bequest motive is $\varepsilon = 1$. The time horizon is set to $T = 10$ years. We begin with numerical examples for the truncated Heston model (i.e., $p = 0.5$ in (8.2)).

Computational Efficiency. The theoretical convergence rate identified in Theorem 7.6 materializes quickly in practice. Typical running times for the solutions reported below are well under 5 seconds.[5] To quantify the convergence speed, Figure 8.1 depicts the logarithmic relative deviations

$$\log_{10}\left(\frac{\|h_n - h_{n-1}\|_\infty}{\|h_{n-1}\|_\infty}\right) \quad \text{and} \quad \log_{10}\left(\frac{\|\frac{\partial}{\partial y}h_n - \frac{\partial}{\partial y}h_{n-1}\|_\infty}{1 + \|\frac{\partial}{\partial y}h_{n-1}\|_\infty}\right) \tag{8.4}$$

as a function of the number of iterations n. Figure 8.1 clearly illustrates the superlinear convergence of our method. Figure 8.2 shows the convergence of

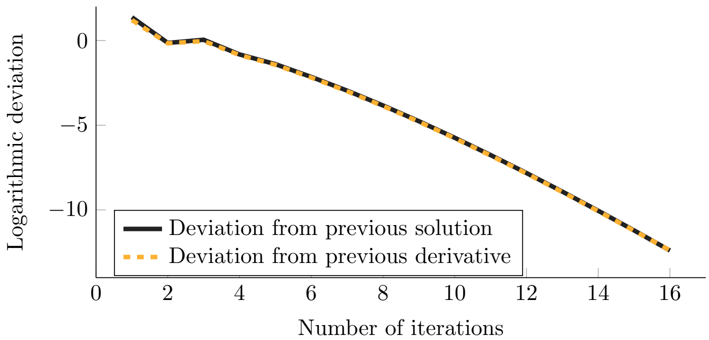

Fig. 8.1 Logarithmic Deviation from Previous Solution. This figure depicts the convergence speed (8.4) of the value function. This figure is based on a truncated Heston model with parameters (8.3).

Algorithm 8.1. We plot the intermediate solutions after $n = 1, 2, \ldots, 5, 10, 15$ steps of the iteration. It is apparent that the algorithm converges quickly: After $n = 5$ steps the solution is visually indiscernible from subsequent iterations; the solutions for $n \geq 15$ are even numerically indistinguishable.

Optimal Strategies. Figure 8.3 illustrates the optimal consumption-wealth ratio $(c/x)^\star$ at time $t = 0$ as a function of initial volatility σ_0 for a risk aversion of $\gamma = 5$ and an EIS of $\psi \in \{0.5, 1, 1.5\}$. For reasonable risk aversions, the optimal stock allocations as a function of σ_0 are almost flat. For instance, for $\gamma \in \{3, 4, 5, 6, 10\}$ and $\psi = 0.5$ the demands vary between about 110% and 30%.

[5] Machine: Intel® Core™ i3-540 Processor (4M Cache, 3.06 GHz), 4 GB RAM.

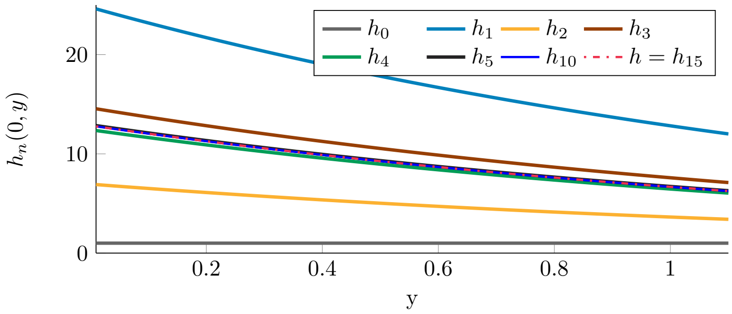

Fig. 8.2 Approximation after n Iteration Steps. The functions h_n described in Algorithm 8.1 converge to the solution h of the reduced HJB equation. This figure is based on a truncated Heston model with parameters (8.3).

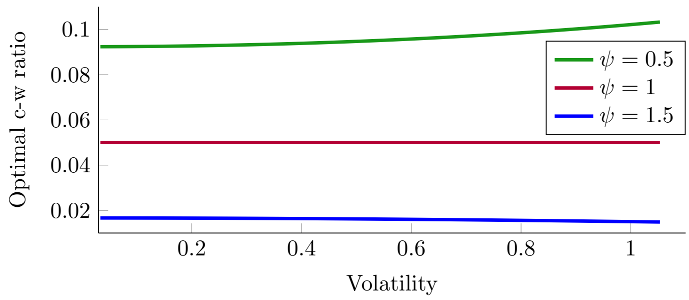

Fig. 8.3 Optimal Consumption-Wealth Ratio. This figure depicts the optimal consumption-wealth ratio $(c/x)^\star$ at time $t = 0$ as a function of initial volatility σ_0 for a risk aversion of $\gamma = 5$. It is based on a truncated Heston model with parameters (8.3).

Comparison with Known Solutions. Figure 8.4 shows a range of solutions of (4.5) as the EIS ψ varies. Here we have chosen $\gamma = 2$ so that for $\psi = 0.125$ (the lowest graph in Figure 8.4) an explicit solution is available (see [26]). For $\psi = 1$ we use the finite-horizon analog of the explicit solution in [12]. The solutions for the other values of the EIS are computed by applying Algorithm 8.1. Note that Figure 8.4 depicts $g \triangleq h^{\frac{k}{1-\gamma}}$ so that the value function can be represented as $w(t, x, y) = \frac{1}{1-\gamma} x^{1-\gamma} h(t, y)^k = \frac{1}{1-\gamma} \left(g(t, y)x\right)^{1-\gamma}$ where in this context g can be interpreted as a cash multiplier.

Finally, we present comparative statics for the model (8.1) where we vary the power p. Figure 8.5 shows the value of the stock demand $\pi^\star$ at time $t = 0$ as a function of the initial volatility σ_0 and the power p. Here $\gamma = 5$ and the EIS is $\psi = 1.5$.

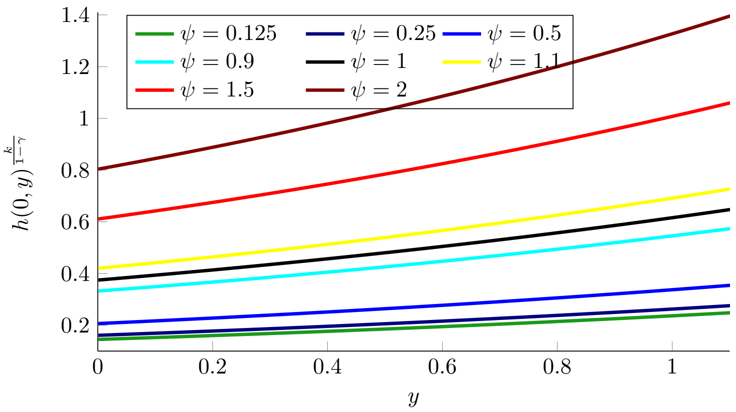

Fig. 8.4 Value Function for Different EIS. This figure compares the function $h^{\frac{k}{1-\gamma}}$ at time $t = 0$ for a risk aversion of $\gamma = 2$ and an EIS of $\psi \in \{0.125, 0.25, 0.5, 0.9, 1, 1.1, 1.5, 2\}$. It is based on a truncated Heston model with parameters (8.3).

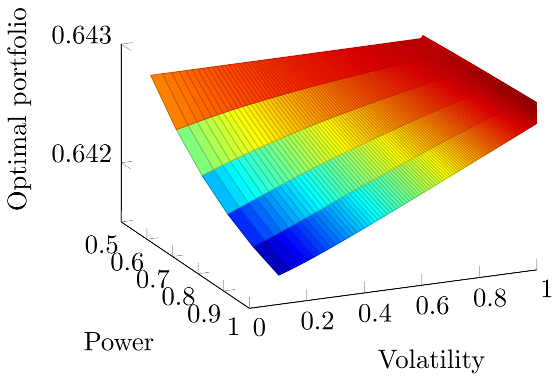

Fig. 8.5 Optimal Stock Demand and Power. This figure depicts the optimal stock demand $\pi^\star$ at time $t = 0$ as a function of initial volatility σ_0 and the power p. The model is (8.1) so $p = 0.5$ corresponds to the Heston model. The calibration is given by parameters (8.3), the agent's risk aversion is $\gamma = 5$ and his EIS is $\psi = 1.5$.

8.3 Exponential Vasicek Model

As another application, we consider the asset price dynamics

$$\mathrm{d}\, S_t = S_t[(r + \bar{\lambda} e^{2I(Y_t)})\,\mathrm{d}\, t + e^{I(Y_t)}\,\mathrm{d}\, W_t]$$

with interest rate $r = 0.05$, $\bar{\lambda} = 3.11$, and I an approximation of the identity function on $\mathbb{R}$. The state process has modified Vasicek dynamics $\mathrm{d}\, Y_t = (\vartheta - \kappa I(Y_t))\,\mathrm{d}\, t + \bar{\beta}(\rho\,\mathrm{d}\, W_t + \sqrt{1-\rho^2}\,\mathrm{d}\, \bar{W}_t)$ with mean reversion speed $\kappa = 5$ and mean reversion level $\vartheta/\kappa = -1.933$. The correlation is set to $\rho = -0.5$ and we put $\bar{\beta} = 0.57$. These parameters are chosen in such a way that the long-term mean and variance of the squared-volatility process coincide with those of the squared volatility process in the Heston model (8.2) calibrated according to (8.3). We continue to use the time preference rate $\delta = 0.05$ and bequest $\varepsilon = 1$.

Figure 8.6 depicts the optimal consumption-wealth ratio at time $t = 0$ as a function of initial volatility for a risk aversion of $\gamma = 5$ and an EIS of $\psi \in \{0.5, 1, 1.5\}$. Figure 8.7 shows optimal stock allocations as a function of

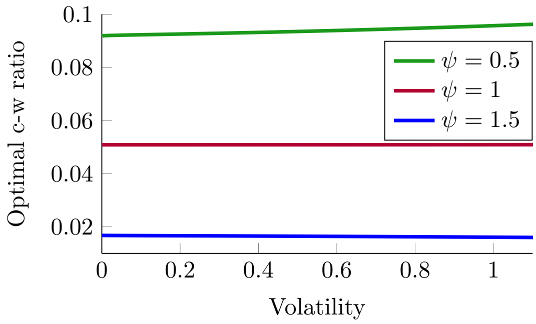

Fig. 8.6 Optimal Consumption-Wealth Ratio. This figure depicts the optimal consumption-wealth ratio $(c/x)^\star$ at time $t = 0$ as a function of initial volatility σ_0 for $\gamma = 5$. This figure is based on a modified exponential Vasicek model with $\kappa = 5$, $\vartheta/\kappa = -1.933$, $\rho = -0.5$, and $\bar{\beta} = 0.57$.

initial volatility for $\gamma \in \{3, 4, 5, 6, 10\}$ and $\psi = 0.5$.

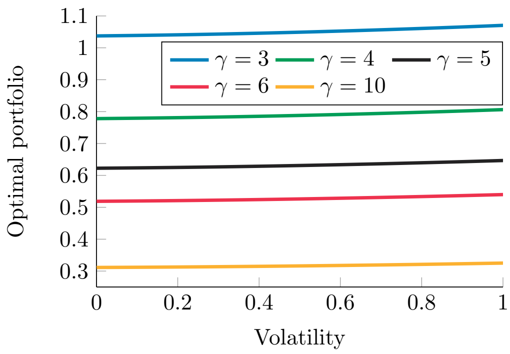

Fig. 8.7 Optimal Stock Demand and Risk Aversion. This figure shows the optimal stock allocation $\pi^\star$ at time $t = 0$ as a function of initial volatility σ_0. The risk aversion varies and the EIS is $\psi = 0.5$. This figure is based on a modified exponential Vasicek model with $\kappa = 5$, $\vartheta/\kappa = -1.933$, $\rho = -0.5$, and $\bar{\beta} = 0.57$.

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A Proofs Omitted from the Main Text

Proof (of Lemma 4.4) Since h solves the reduced HJB equation (4.5) we have

$$H(z,\hat{\pi},\hat{c}) \triangleq w_t + x(r+\hat{\pi}\lambda)w_x - \hat{c}w_x + \tfrac{1}{2}x^2\hat{\pi}^2\sigma^2 w_{xx} + \alpha w_y + \tfrac{1}{2}\beta^2 w_{yy} + x\hat{\pi}\sigma\beta\rho w_{xy} + f(\hat{c},w) = 0$$

where $z \triangleq (t,x,y,w_x,w_y,w_{x_y}w_{xx},w_{yy})$. Separating $H(z,\pi,c) \triangleq u(z,\pi) + s(z,c) + q(z)$, it is easy to see that the candidate solutions $\hat{\pi}$ and $\hat{c}$ defined in (4.4) are the unique solutions of the associated first-order conditions

$$\begin{aligned} 0 &= s_c(z,c) = -w_x + f_c(c,w), \\ 0 &= u_\pi(z,\pi) = x\lambda w_x + \pi x^2\sigma^2 w_{xx} + x\sigma\beta\rho w_{xy}. \end{aligned} \tag{A.1}$$

Concavity of u and s implies that $H(z,\hat{\pi},\hat{c}) = \sup_{\pi\in\mathbb{R},\, c\in(0,\infty)} H(z,\pi,c)$. □

Proof (of Lemma 4.5) By (A1) and (A2), $\tilde{\alpha}$ and $\tilde{r}$ are bounded. Moreover

$$|\tilde{\alpha}(y)-\tilde{\alpha}(\bar{y})| \le |\tfrac{1-\gamma}{\gamma}|\rho\left(|\tfrac{\lambda(y)}{\sigma(y)}||\beta(y)-\beta(\bar{y})| + |\tfrac{\beta(\bar{y})}{\sigma(y)}||\lambda(y)-\lambda(\bar{y})|\right) + |\beta(\bar{y})\lambda(\bar{y})|\tfrac{\sigma(\bar{y})-\sigma(y)}{\sigma(y)\sigma(\bar{y})}| + |\alpha(y)-\alpha(\bar{y})|$$

so $\tilde{\alpha}$ is Lipschitz continuous. Finally,

$$k|\tilde{r}(y)-\tilde{r}(\bar{y})| \le |1-\gamma||r(y)-r(\bar{y})| + |\tfrac{1-\gamma}{\gamma}|\|\lambda\|_\infty(\inf_{x\in\mathbb{R}}\sigma(x))^{-2}|\lambda(y)-\lambda(\bar{y})| + |\tfrac{1-\gamma}{\gamma}|\|\lambda\|_\infty^2\|\sigma\|_\infty(\inf_{x\in\mathbb{R}}\sigma(x))^{-4}|\sigma(\bar{y})-\sigma(y)|.$$ □

Proof (of Lemma 5.3) The candidate optimal wealth process $\hat{X}$ has dynamics

$$\mathrm{d}\hat{X}_t = \hat{X}_t\left[(r_t + \tfrac{1}{\gamma}\tfrac{\lambda_t^2}{\sigma_t^2} + \tfrac{k}{\gamma}\tfrac{\lambda_t\beta_t\rho}{\sigma_t}\tfrac{h_y}{h} - \delta^\psi h^{q-1})\,\mathrm{d}t + (\tfrac{1}{\gamma}\tfrac{\lambda_t}{\sigma_t} + \tfrac{k}{\gamma}\beta_t\rho\tfrac{h_y}{h})\,\mathrm{d}W_t\right].$$

Put $a_t \triangleq r_t + \frac{1}{\gamma}\frac{\lambda_t^2}{\sigma_t^2} + \frac{k}{\gamma}\frac{\lambda_t\beta_t\rho}{\sigma_t}\frac{h_y}{h} - \delta^\psi h^{q-1}$ and $b_t \triangleq \frac{1}{\gamma}\frac{\lambda_t}{\sigma_t} + \frac{k}{\gamma}\beta_t\rho\frac{h_y}{h}$. Our assumptions on the coefficients and on h_y and h imply that both a and b are bounded. By Itō's formula

$$\hat{X}_t^p = x^p\exp\left(p\int_0^t\left(a_s + \tfrac{1}{2}(p-1)b_s^2\right)\mathrm{d}s\right)\mathcal{E}_t\left(p\int_0^\cdot b_s\,\mathrm{d}W_s\right)$$

where $\mathcal{E}_t(\cdot)$ denotes the stochastic exponential. Choose $M > 0$ such that $|pa_t| + |p(p-1)b_t^2|, |pb_t| < M$ for all $t \in [0,T]$. By Novikov's condition $\mathcal{E}_t\left(p\int_0^\cdot b_s \,\mathrm{d}\,W_s\right)$ is an L^2-martingale, so using Doob's L^2-inequality we get

$$\mathrm{E}[\sup_{t\in[0,T]}\hat{X}_t^p] \leq 2x^p e^{MT}\,\mathrm{E}[\mathcal{E}_T\left(p\textstyle\int_0^\cdot b_s \,\mathrm{d}\,W_s\right)^2]^{\frac{1}{2}} < \infty. \qquad \square$$

Proof (of Lemma 5.4) Lemma 5.3 and the boundedness of $\delta^\psi h(t,Y_t)^{q-1}$ imply that $\mathrm{E}[\sup_{t\in[0,T]}|\hat{c}_t|^p] < \infty$ for all $p \in \mathbb{R}$. In particular $\hat{c} \in \mathcal{C}$. By Itō's formula

$$\begin{aligned}\mathrm{d}\,V_t = \big[&w_t + \hat{X}_t(r_t + \hat{\pi}_t\lambda_t)w_x - \hat{c}_t w_x + \tfrac{1}{2}\hat{X}_t^2\hat{\pi}_t^2\sigma_t^2 w_{xx} + \alpha_t w_y + \tfrac{1}{2}\beta_t^2 w_{yy}\\ &+\hat{X}_t\hat{\pi}_t\sigma_t\beta_t\rho w_{xy}\big]\,\mathrm{d}\,t + \mathrm{d}\,M_t\end{aligned}$$

where M is a local martingale. Hence $\mathrm{d}\,V_t = -f(\hat{c}_t, V_t)\,\mathrm{d}\,t + \mathrm{d}\,M_t$ by Lemma 4.4. Moreover, exploiting the special form of w we get

$$\mathrm{d}\,M_t = V_t\left[\tfrac{1-\gamma}{\gamma}\tfrac{\lambda_t}{\sigma_t} + \tfrac{\rho k}{\gamma}\beta_t\tfrac{h_y}{h}\right]\mathrm{d}\,W_t + V_t k\sqrt{1-\rho^2}\beta_t\tfrac{h_y}{h}\,\mathrm{d}\,\bar{W}_t.$$

Here V_t can be rewritten as $V_t = w(t,\hat{X}_t,Y_t) = \frac{1}{1-\gamma}\hat{X}_t^{1-\gamma}h(t,Y_t)^k$. By (4.6) the function h is bounded and bounded away from zero. Thus $\mathrm{E}[\sup_{t\in[0,T]}|V_t|^p] < \infty$ for all $p \in \mathbb{R}$, by Lemma 5.3. Hence V is a utility process associated with $\hat{c}$; by (E1) it follows that $V = V^{\hat{c}}$. Finally, the first-order condition (A.1) for optimal consumption implies $w_x(t,\hat{X}_t,Y_t) = f_c(t, w(t,\hat{X}_t,Y_t)) = f_c(\hat{c}_t,\hat{V}_t)$.

$\square$

Proof (of Lemma 5.5) For simplicity of notation we set $r_t \triangleq r(Y_t)$, $\lambda_t \triangleq \lambda(Y_t)$ and $\sigma_t \triangleq \sigma(Y_t)$. We have $\mathrm{d}\,Z_t^{\pi,c} = \hat{m}_t c_t\,\mathrm{d}\,t + \hat{m}_t\,\mathrm{d}\,X_t^{\pi,c} + X_t^{\pi,c}\,\mathrm{d}\,\hat{m}_t + \mathrm{d}[\hat{m}_t, X_t^{\pi,c}]$, by the product rule. Inserting the dynamics of $X^{\pi,c}$ from (4.1) we get

$$\mathrm{d}\,Z_t^{\pi,c} = \hat{m}_t X_t^{\pi,c}[(r_t + \pi_t\lambda_t)\,\mathrm{d}\,t + \pi_t\sigma_t\,\mathrm{d}\,W_t] + X_t^{\pi,c}\,\mathrm{d}\,\hat{m}_t + \mathrm{d}[\hat{m}_t, X_t^{\pi,c}].$$

By Lemma 5.4, $\hat{V}_t = w(t,\hat{X}_t,Y_t)$ and $\hat{m}_t = e^{\int_0^t f_v(\hat{c}_s,\hat{V}_s)\,\mathrm{d}\,s} w_x(t,\hat{X}_t,Y_t)$. From here on we abbreviate $f_v = f_v(\hat{c}_t,\hat{V}_t)$, $w_x = w_x(t,\hat{X}_t,Y_t)$ etc. Clearly $\mathrm{d}\,\hat{m}_t = \hat{m}_t[f_v\,\mathrm{d}\,t + \frac{\mathrm{d}\,w_x}{w_x}]$. Since $f_v(c,v) = \delta\frac{\phi-\gamma}{1-\phi}c^{1-\phi}[(1-\gamma)v]^{\frac{\phi-1}{1-\gamma}} - \delta\theta$, we obtain that $f_v(\hat{c}_t, w(t,\hat{X}_t,Y_t)) = \frac{\phi-\gamma}{1-\phi}\delta^\psi h^{q-1} - \delta\theta$. By Itō's formula

$$\mathrm{d}\,w_x = w_x\left[\tfrac{w_{xt}}{w_x}\,\mathrm{d}\,t + \tfrac{w_{xx}}{w_x}\,\mathrm{d}\,\hat{X}_t + \tfrac{1}{2}\tfrac{w_{xxx}}{w_x}\,\mathrm{d}[\hat{X}_t] + \tfrac{1}{2}\tfrac{w_{xyy}}{w_x}\,\mathrm{d}[Y_t] + \tfrac{w_{xxy}}{w_x}\,\mathrm{d}[\hat{X}_t,Y_t]\right].$$

Substituting for w we find

$$\begin{aligned}\tfrac{\mathrm{d}\,w_x}{kw_x} &= \tfrac{h_t}{h}\,\mathrm{d}\,t - \tfrac{\gamma}{k}\tfrac{\mathrm{d}\,\hat{X}_t}{\hat{X}_t} + \tfrac{h_y}{h}\,\mathrm{d}\,Y_t + \tfrac{1}{2}\tfrac{\gamma(1+\gamma)}{k}\tfrac{\mathrm{d}[\hat{X}_t]}{\hat{X}_t^2}\\ &\quad+\tfrac{1}{2}\left((k-1)\tfrac{h_y^2}{h^2} + \tfrac{h_{yy}}{h}\right)\mathrm{d}[Y_t] - \tfrac{\gamma}{\hat{X}_t}\tfrac{h_y}{h}\,\mathrm{d}[\hat{X}_t,Y_t].\end{aligned}$$

Plugging in the candidate $\hat{\pi}$ from (5.1) and the dynamics of $\hat{X}$ and Y yields

$$\tfrac{\mathrm{d}\,w_x}{kw_x} = A_t^1\,\mathrm{d}\,t + A_t^2\,\mathrm{d}\,t - \tfrac{1}{k}\tfrac{\lambda_t}{\sigma_t}\,\mathrm{d}\,W_t + \sqrt{1-\rho^2}\beta_t\tfrac{h_y}{h}\,\mathrm{d}\,\bar{W}_t, \quad \text{where}$$

$$A_t^1 \triangleq \frac{h_t}{h} - \frac{\gamma}{k} r_t + \frac{1}{2}\frac{1}{k}\frac{1-\gamma}{\gamma}\frac{\lambda_t^2}{\sigma_t^2} + \frac{1}{\gamma}\frac{\lambda_t\beta_t\rho}{\sigma}\frac{h_y}{h} + \frac{\gamma}{k}\delta^\psi h^{q-1} + \frac{k}{2}\frac{1+\gamma}{\gamma}\beta_t^2\rho^2\frac{h_y^2}{h^2}$$
$$A_t^2 \triangleq \frac{h_y}{h}\left(\alpha_t - \frac{\rho\beta_t\lambda_t}{\sigma_t}\right) + \frac{h_y^2}{h^2}\left(\frac{k-1}{2}\beta_t^2 - k\beta_t^2\rho^2\right) + \frac{\beta_t^2}{2}\frac{h_{yy}}{h}.$$

For the sum of the $\frac{h_y^2}{h^2}$-terms we have

$$\frac{k}{2}\frac{1+\gamma}{\gamma}\beta_t^2\rho^2\frac{h_y^2}{h^2} + \frac{h_y^2}{h^2}\left(\frac{k-1}{2}\beta_t^2 - k\beta_t^2\rho^2\right) = \beta_t^2\frac{h_y^2}{h^2}\left(\frac{k}{2}\rho^2\frac{1+\gamma}{\gamma} + \frac{k-1}{2} - \rho^2 k\right) = 0$$

by our choice of k. Combining the above we obtain

$$\begin{aligned}\mathrm{d}\,\hat{m}_t = k\hat{m}_t &\left[\frac{h_t}{h} + \frac{1}{k}\left(-\gamma r_t + \frac{1}{2}\frac{1-\gamma}{\gamma}\frac{\lambda_t^2}{\sigma_t^2} - \delta\theta\right) + \tilde{\alpha}_t\frac{h_y}{h} + \frac{\beta_t^2}{2}\frac{h_{yy}}{h} + \frac{\phi\theta}{k}\delta^\psi h^{q-1}\right] \\ &+ k\hat{m}_t\left[-\frac{1}{k}\frac{\lambda_t}{\sigma_t}\,\mathrm{d}\,W_t + \sqrt{1-\rho^2}\beta_t\frac{h_y}{h}\,\mathrm{d}\,\bar{W}_t\right]\end{aligned}$$

and it follows that $\mathrm{d}[\hat{m}_t, X_t^{\pi,c}] = -\lambda_t\pi_t\hat{m}_t X_t^{\pi,c}\,\mathrm{d}\,t$. Since h solves (4.5) we get

$$\begin{aligned}\mathrm{d}\,Z_t^{\pi,c} &= \hat{m}_t X_t^{\pi,c}[(r_t + \pi_t\lambda_t)\,\mathrm{d}\,t + \pi_t\sigma_t\,\mathrm{d}\,W_t] + X_t^{\pi,c}\,\mathrm{d}\,\hat{m}_t + \mathrm{d}[\hat{m}_t, X_t^{\pi,c}] \\ &= \hat{m}_t X_t^{\pi,c}\frac{1}{h}\left[h_t - \tilde{r}_t h + \tilde{\alpha}_t h_y + \frac{1}{2}\beta_t^2 h_{yy} + \frac{\delta^\psi}{1-q}h^q\right]\mathrm{d}\,t + \mathrm{d}\,M_t = \mathrm{d}\,M_t\end{aligned}$$

where $\mathrm{d}\,M_t \triangleq \hat{m}_t X_t^{\pi,c}[(\pi_t\sigma_t - \frac{\lambda_t}{\sigma_t})\,\mathrm{d}\,W_t + k\sqrt{1-\rho^2}\beta_t\frac{h_y}{h}\,\mathrm{d}\,\bar{W}_t]$ defines a local martingale M. A direct calculation using the definition of $\hat{\pi}$ yields the statement for $Z^{\hat{\pi},\hat{c}}$. □

Proof (of Lemma 5.6) Recall that $\underline{h} \le h \le \overline{h}$ so

$$f_v(\hat{c}_s, \hat{V}_s) = \frac{\phi-\gamma}{1-\phi}\delta^\psi h(s, Y_s)^{q-1} - \delta\theta \le \left|\frac{\phi-\gamma}{1-\phi}\right|\delta^\psi\left(\underline{h}^{q-1} + \overline{h}^{q-1}\right) + |\delta\theta| \triangleq m_1$$

and we get $0 \le \exp(p\int_0^T f_v(\hat{c}_s, \hat{V}_s)\,\mathrm{d}\,s) \le e^{Tpm_1}$. On the other hand, Lemma 5.4 implies $\mathrm{E}[\sup_{t\in[0,T]} f_c(\hat{c}_t, \hat{V}_t)^p] < \infty$ for all $p \in \mathbb{R}$. It follows that

$$\mathrm{E}\big[\sup_{t\in[0,T]}\hat{m}_t^p\big] < \infty \quad \text{for all } p > 1.$$

To show that $Z^{\hat{\pi},\hat{c}}$ is a martingale, note that $\frac{1-\gamma}{\gamma}\frac{\lambda_t}{\sigma_t} + \frac{k}{\gamma}\beta_t\rho\frac{h_y}{h}$ is uniformly bounded by some $c > 0$. Hence by Lemma 5.3 we have

$$\int_0^T \mathrm{E}\left[\hat{m}_t^2\hat{X}_t^2\left(\frac{1-\gamma}{\gamma}\frac{\lambda_t}{\sigma_t} + \frac{k}{\gamma}\beta_t\rho\frac{h_y}{h}\right)^2\right]\mathrm{d}\,t \le c^2\int_0^T\sqrt{\mathrm{E}[\hat{m}_t^4]\,\mathrm{E}[\hat{X}_t^4]}\,\mathrm{d}\,t < \infty.$$

Analogously we obtain $\int_0^T \mathrm{E}[\hat{m}_t^2\hat{X}_t^2(k\sqrt{1-\rho^2}\beta_t\frac{h_y}{h})^2]\,\mathrm{d}\,t < \infty$. From this and Lemma 5.5, we conclude that $Z^{\hat{\pi},\hat{c}}$ is an L^2-martingale. □

Proof (of Proposition 6.2) For any fixed $\kappa > c+\varrho$, define a metric d equivalent to $\|\cdot\|_\infty$ by $d(X,Y) \triangleq \operatorname{ess\,sup}_{\mathrm{d}\,t\otimes P} e^{-\kappa(T-t)}|X_t - Y_t|$. Then (A,d) is a complete metric space. By definition $|X_s - Y_s| \le e^{\kappa(T-s)} d(X,Y)$ $\mathrm{d}\,t \otimes P$-a.e., so

$$\begin{aligned} e^{-\kappa(T-t)}|(SX)_t - (SY)_t| &\le e^{-\kappa(T-t)} c \int_t^T e^{(s-t)\varrho} e^{\kappa(T-s)} d(X,Y)\,\mathrm{d}\,s \\ &\le \tfrac{c}{\kappa-\varrho} d(X,Y) \end{aligned}$$

and we conclude that $d(SX,SY) \le \frac{c}{\kappa-\varrho} d(X,Y)$, where $\frac{c}{\kappa-\varrho} < 1$. Hence S is a contraction on (A,d). Thus by Banach's Fixed Point Theorem there is a unique $X \in A$ with $SX = X$, and for all $n \in \mathbb{N}$ we have $d(X_{(n)}, X) \le (\frac{c}{\kappa-\varrho})^n d(X_{(0)}, X)$. Hence it follows that

$$\begin{aligned} |(X_{(n)})_t - X_t| &\le e^{\kappa T} d(X_{(n)}, X) \le (\tfrac{c}{\kappa-\varrho})^n e^{\kappa T} d(X_{(0)}, X) \\ &\le e^{\kappa T}(\|X_{(0)}\|_\infty + \|X\|_\infty)(\tfrac{c}{\kappa-\varrho})^n \end{aligned}$$

and thus $\|X_{(n)} - X\|_\infty \le e^{\kappa T}(\|X_{(0)}\|_\infty + \|X\|_\infty)(\frac{c}{\kappa-\varrho})^n$, for every $n \in \mathbb{N}$ and every choice of $\kappa > c+\varrho$. Setting $\kappa = \frac{n+T\varrho}{T}$ for $n > cT$ we obtain the asserted error bound. □

B Stochastic Gronwall Inequality

This appendix provides the ramification of the stochastic Gronwall-Bellman inequality required for the proofs in this article. Related results can be found in [17], [1], and [36]. We work on a general probability space $(\Omega, \mathcal{F}, P)$ that is endowed with a filtration $(\mathcal{F}_t)_{t\ge 0}$ that is right-continuous and complete.

Proposition B.1 (Stochastic Gronwall-Bellman Inequality) *Let $A = (A_t)_{t\in[0,T]}$ be bounded and progressively measurable, $Z \in L^p(P)$ and let $B = (B_t)_{t\in[0,T]}$ be a progressively measurable process in $L^p(\mathrm{d}\,t \otimes P)$ for some $p > 1$. Moreover let $X = (X_t)_{t\in[0,T]}$ be right-continuous and adapted with $\mathrm{E}[\sup_{t\in[0,T]} |X_t|] < \infty$. If*

$$1_{\{\tau>t\}} X_t \ge \mathrm{E}_t\left[1_{\{\tau>t\}} \int_t^\tau (A_s X_s + B_s)\,\mathrm{d}\,s + 1_{\{\tau>t\}} X_\tau\right] \quad \text{a.s. for } t \in [0,T] \tag{B.1}$$

for every stopping time τ and $X_T \ge Z$, then

$$X_t \ge \mathrm{E}_t\left[\int_t^T e^{\int_t^s A_u\,\mathrm{d}\,u} B_s\,\mathrm{d}\,s + e^{\int_t^T A_s\,\mathrm{d}\,s} Z\right] \quad \text{for all } t \in [0,T] \text{ a.s.}$$

Proof We set

$$M_t \triangleq \mathrm{E}_t\left[\int_0^T e^{\int_0^s A_u\,\mathrm{d}\,u} B_s\,\mathrm{d}\,s + e^{\int_0^T A_s\,\mathrm{d}\,s} Z\right].$$

Since A is bounded above, $Z \in L^p(P)$ and $B \in L^p(\mathrm{d}\,t \otimes P)$, it follows from Doob's L^p-inequality that $\mathrm{E}[\sup_{t\in[0,T]} |M_t|^p] < \infty$. In particular M is well-defined as a uniformly integrable martingale. Now set

$$Y_t \triangleq e^{-\int_0^t A_s \,\mathrm{d}\, s} \left(M_t - \int_0^t e^{\int_0^s A_u \,\mathrm{d}\, u} B_s \,\mathrm{d}\, s \right).$$

Since A is bounded below we have $\mathrm{E}[\sup_{t\in[0,T]} |Y_t|^p] < \infty$, and integration by parts yields

$$\begin{aligned} \mathrm{d}\, Y_t &= e^{-\int_0^t A_s \,\mathrm{d}\, s} \left(\mathrm{d}\, M_t - e^{\int_0^t A_u \,\mathrm{d}\, u} B_t \,\mathrm{d}\, t \right) - Y_{t-} A_t \,\mathrm{d}\, t \\ &= - \left[A_t Y_t + B_t \right] \mathrm{d}\, t + \mathrm{d}\, N_t \end{aligned}$$

where $N_t \triangleq \int_0^t e^{-\int_0^s A_u \,\mathrm{d}\, u} \,\mathrm{d}\, M_s$ is a uniformly integrable martingale. For an arbitrary stopping time τ we obtain

$$1_{\{\tau>t\}}(Y_t - Y_\tau) = 1_{\{\tau>t\}} \left(-\int_0^t (A_s Y_s + B_s) \,\mathrm{d}\, s + N_t + \int_0^\tau (A_s Y_s + B_s) \,\mathrm{d}\, s - N_\tau \right)$$

so

$$1_{\{\tau>t\}} Y_t = 1_{\{\tau>t\}} \int_t^\tau (A_s Y_s + B_s) \,\mathrm{d}\, s + 1_{\{\tau>t\}}(N_t - N_\tau) + 1_{\{\tau>t\}} Y_\tau.$$

Since $(1_{\{\tau>t\}}(N_t - N_\tau))_{s\in[t,T]}$ is a martingale, it follows that

$$1_{\{\tau>t\}} Y_t = \mathrm{E}_t \left[1_{\{\tau>t\}} \int_t^\tau (A_s Y_s + B_s) \,\mathrm{d}\, s + 1_{\{\tau>t\}} Y_\tau \right]. \tag{B.2}$$

We set $\Delta_t \triangleq X_t - Y_t$ and obtain $\Delta_T = X_T - Z \geq 0$ and $\mathrm{E}[\sup_{t\in[0,T]} |\Delta_t|] < \infty$. Moreover (B.1) and (B.2) imply that, for any stopping time τ,

$$1_{\{\tau>t\}} \Delta_t \geq \mathrm{E}_t \left[1_{\{\tau>t\}} \int_t^\tau A_s \Delta_s \,\mathrm{d}\, s + 1_{\{\tau>t\}} \Delta_\tau \right] \quad \text{a.s. for all } t \in [0,T].$$

Thus Proposition C.2 in [40] applies to yield $\Delta_t \geq 0$ for all $t \in [0,T]$ a.s., i.e.

$$X_t \geq Y_t = e^{-\int_0^t A_u \,\mathrm{d}\, u} \,\mathrm{E}_t \left[\int_t^T e^{\int_0^s A_u \,\mathrm{d}\, u} B_s \,\mathrm{d}\, s + e^{\int_0^T A_s \,\mathrm{d}\, s} Z \right]. \qquad \square$$

C Some Facts on Parabolic Partial Differential Equations

This appendix collects the relevant results on linear and semilinear parabolic partial differential equations that are used in this article. Following [29] we first introduce the Hölder spaces $H^{r/2,r}([0,T] \times \mathbb{R}^d)$ for $r \in \mathbb{R}^+$. For a continuous function $u : [0,T] \times \mathbb{R}^d \to \mathbb{R}, (t,x) \mapsto u(t,x)$ and $q \in (0,1)$ we define the Hölder coefficient $\langle u \rangle_x^q$ in space via

$$\langle u \rangle_x^q \triangleq \sup_{t\in[0,T],\ x,x'\in\mathbb{R}^d,\ |x-x'|\leq 1} \frac{|u(t,x) - u(t,x')|}{|x - x'|^q}$$

and the Hölder coefficient $\langle u\rangle_t^q$ in time via

$$\langle u\rangle_t^q \triangleq \sup_{t,t'\in[0,T],\ x\in\mathbb{R}^d,\ |t-t'|\le 1} \frac{|u(t,x)-u(t,x')|}{|t-t'|^q}\,.$$

The space $H^{r/2,r}([0,T]\times\mathbb{R}^d)$ consists of all functions $u:[0,T]\times\mathbb{R}^d\to\mathbb{R}$ that are continuous along with all derivatives $D_t^\alpha D_x^\beta u$ with "order" $2|\alpha|+|\beta|\le r$ and that satisfy $\|u\|_H^{r/2,r}<\infty$. Here the norm $\|u\|_H^{r/2,r}$ of u is given by

$$\|u\|_H^{r/2,r} \triangleq \langle u\rangle_\bullet^{r/2,r} + \sum_{2|\alpha|+|\beta|\le\lfloor r\rfloor} \|D_t^\alpha D_x^\beta u\|_\infty$$

where the mixed space-time Hölder coefficient $\langle u\rangle_\bullet^{r/2,r}$ of u is given by

$$\langle u\rangle_\bullet^{r/2,r} \triangleq \sum_{2|\alpha|+|\beta|=\lfloor r\rfloor} \langle D_t^\alpha D_x^\beta u\rangle_x^{r-\lfloor r\rfloor} + \sum_{r-2<2|\alpha|+|\beta|<r} \langle D_t^\alpha D_x^\beta u\rangle_t^{\frac{r-2|\alpha|-|\beta|}{2}}\,.$$

Thus $\|u\|_H^{r/2,r}$ sums up the L^∞-norms of all relevant derivatives, plus the Hölder coefficients of the highest-order derivatives. Analogously, for $r\in\mathbb{R}^+$ the space $H^r(\mathbb{R}^d)$ is defined as the collection of all $\lfloor r\rfloor$-times continuously differentiable functions $u:\mathbb{R}^d\to\mathbb{R}$ with $\|u\|_H^r<\infty$ where[6]

$$\|u\|_H^r \triangleq \langle u\rangle_\bullet^r + \sum_{|\beta|\le\lfloor r\rfloor}\|D^\beta u\|_\infty \quad\text{and}\quad \langle u\rangle_\bullet^r \triangleq \sum_{|\beta|=\lfloor r\rfloor}\langle D^\beta u\rangle^{r-\lfloor r\rfloor}\,.$$

Linear Cauchy Problem. Consider a linear second-order differential operator

$$Lu \triangleq \tfrac{\partial u}{\partial t} - \textstyle\sum_{i,j=1}^d a_{ij}(t,x)\tfrac{\partial^2 u}{\partial x_i\partial x_j} - \sum_{i=1}^d b_i(t,x)\tfrac{\partial u}{\partial x_i} - c(t,x)u$$

where the coefficients a,b,c are defined on $[0,T]\times\mathbb{R}^d$ and $(a_{i,j}(t,x))_{i,j}$ is a symmetric matrix for all $(t,x)\in[0,T]\times\mathbb{R}^d$. The main existence and uniqueness result for linear Cauchy problems in $\mathbb{R}^d$, Theorem C.1 below, relies on the following conditions:

(P1) The operator L is uniformly parabolic, i.e. there exist $0<c_1<c_2<\infty$ such that for every $(t,x)\in[0,T]\times\mathbb{R}^d$ we have

$$c_1|y|^2 \le \textstyle\sum_{i,j=1}^d a_{ij}(t,x)y_iy_j \le c_2|y|^2 \quad\text{for all } y\in\mathbb{R}^d.$$

$(\mathrm{P2})^r$ The coefficients satisfy $a_{i,j},b_i,c\in H^{r/2,r}([0,T]\times\mathbb{R}^d)$ for all $i,j=1,\ldots,d$.

[6] Here we slightly abuse notation since $\langle u\rangle_x^q$ has only been defined for functions on $[0,T]\times\mathbb{R}^d$. Of course, for $u:\ \mathbb{R}^d\to\mathbb{R}$ and $q\in(0,1)$ we understand that $\langle u\rangle_x^q \triangleq \sup_{x,x'\in\mathbb{R}^d,\ |x-x'|\le 1}\frac{|u(x)-u(x')|}{|x-x'|^q}$.

Theorem C.1 (Linear Cauchy Problems) *Suppose* (P1) *and* $(\text{P2})^r$ *are satisfied with* $r \in \mathbb{R}^+$, $r \notin \mathbb{N}$, *and let* $\varphi \in H^{r+2}(\mathbb{R}^d)$ *and* $f \in H^{r/2,r}([0,T]\times\mathbb{R}^d)$. *Then there exists a unique* $u \in H^{(r+2)/2,r+2}([0,T] \times \mathbb{R}^d)$ *such that*

$$Lu = f, \quad u(0,\,\cdot\,) = \varphi.$$

Moreover u satisfies

$$\|u\|_H^{r/2+1,r+2} \le c\left(\|\varphi\|_H^{r+2} + \|f\|_H^{r/2,r}\right)$$

where $c > 0$ *is a global constant that is independent of* φ *and* f.

Proof See Theorem 5.1 in [29], p. 320. □

As a special case we obtain the result we have used in the proof of Lemma 7.1:

Corollary C.2 *Suppose that*

(C1) $a, b, c : \mathbb{R} \to \mathbb{R}$ *are bounded and Lipschitz continuous,*
(C2) *the function a has a bounded Lipschitz continuous derivative and satisfies* $\inf_{y\in\mathbb{R}} a(y) > 0$,
(C3') $\hat{\varepsilon} \in H^{r+2}(\mathbb{R})$ *for some* $r \in (0,1)$.

Then for each bounded and Lipschitz continuous function $f : [0,T] \times \mathbb{R} \to \mathbb{R}$ *there exists a unique* $g \in C_b^{1,2}([0,T] \times \mathbb{R})$ *that solves*

$$0 = g_t + ag_{yy} + bg_y + cg + f, \quad g(T,\,\cdot\,) = \hat{\varepsilon}.$$

Proof Consider the second-order differential operator

$$Lu = \tfrac{\partial u}{\partial t} - a\tfrac{\partial^2 u}{\partial y \partial y} - b\tfrac{\partial u}{\partial y} - cu.$$

By assumption (C1) and (C2), the differential operator L satisfies (P1) and $(\text{P2})^r$ for $r \in (0,1)$. Moreover, $f \in H^{r/2,r}([0,T] \times \mathbb{R})$, since f is Lipschitz continuous. Hence Theorem C.1 yields $u \in H^{(r+2)/2,r+2}([0,T] \times \mathbb{R})$ such that

$$Lu = f(T-t,\cdot), \quad u(0,\,\cdot\,) = \hat{\varepsilon} \quad \text{and} \quad \|u\|_{C^{1,2}} \le \|u\|_H^{(r+2)/2,r+2} < \infty.$$

Thus defining $g \in C_b^{1,2}([0,T] \times \mathbb{R})$ by $g(t,y) \triangleq u(T-t,y)$ we obtain

$$0 = g_t + ag_{yy} + bg_y + cg + f, \quad g(T,\cdot) = \hat{\varepsilon}. \qquad \square$$

Quasilinear Cauchy Problem. Next consider the nonlinear differential operator

$$Lu \triangleq u_t - \sum_{i=1}^d \left(\frac{\mathrm{d}}{\mathrm{d}\,x_i} a_i(t,x,u,u_x) \right) + a(t,x,u,u_x)$$

with principal part in divergence form. We set

$$a_{ij}(t,x,u,p) \triangleq \frac{\partial a_i(x,t,u,p)}{\partial p_j} \qquad \text{and} \tag{C.1}$$

$$A(t,x,u,p) \triangleq a(t,x,u,p) - \sum_{i=1}^d \left(\frac{\partial a_i}{\partial u} p_i + \frac{\partial a_i}{\partial x_i} \right).$$

We now state the conditions required for the main result in [29].

(Q1) For all $t \in (0,T]$, $x,p \in \mathbb{R}^d$ and $u \in \mathbb{R}$ we have

$$\sum_{i,j=1}^d a_{ij}(t,x,u,p) y_i y_j \geq 0 \quad \text{for all } y \in \mathbb{R}^d.$$

(Q2) There exist $b_1, b_2 \geq 0$ such that for all $t \in (0,T]$, $x \in \mathbb{R}^d$ and $u \in \mathbb{R}$ we have

$$A(t,x,u,0) \geq -b_1 u^2 - b_2.$$

(Q3) The functions a and a_i are continuous, and a_i is differentiable with respect to x, u, and p. Moreover there exist $c_1, c_2 > 0$ such that for all $v = (t,x,u,p) \in [0,T] \times \mathbb{R}^d \times \mathbb{R} \times \mathbb{R}^d$ we have

$$c_1 |y|^2 \leq \sum_{i,j=1}^n a_{ij}(v) y_i y_j \leq c_2 |y|^2 \quad \text{for all } y \in \mathbb{R}^d$$

and, with a_{ij} given by (C.1),

$$|a(v)| + \sum_{i=1}^d \left(|a_i(v)| + |\frac{\partial a_i(v)}{\partial u} \right) (1+|p|) + \sum_{i,j=1}^d |a_{ij}(v)|$$
$$\leq c_2 (1+|u|)(1+|p|)^2.$$

$(\mathrm{Q4})^\beta$ There exists $\beta \in (0,1)$ such that for all compact sets $K \subset \mathbb{R}$, $\bar{K} \subset \mathbb{R}^d$ the functions

$$a_i, a, a_{i,j}, \frac{\partial a_i}{\partial u}, \frac{\partial a_i}{\partial x_i} : [0,T] \times \mathbb{R}^d \times K \times \bar{K} \to \mathbb{R}$$

are Hölder continuous in t,x,u and p with exponents $\frac{\beta}{2}$, β, β and β, respectively.

Here we say that $f : [0,T] \times \mathbb{R}^d \times K \times \bar{K} \to \mathbb{R}, z = (z^1, z^2, z^3, z^4) \mapsto f(z)$ is β-Hölder continuous in z^i if

$$\langle u \rangle_i^\beta \triangleq \sup_{z,\bar{z} \in \mathrm{Dom}(f),\ z^j = \bar{z}^j,\ j \neq i,\ |z^i - \bar{z}^i| \leq 1} \frac{|f(z) - f(\bar{z})|}{|z^i - \bar{z}^i|^\beta} < \infty.$$

Theorem C.3 (Quasilinear Cauchy Problems) *Suppose* $\psi_0 \in H^{2+\beta}(\mathbb{R}^d)$ *and that* (Q1), (Q2), (Q3) *and* $(\mathrm{Q4})^\beta$ *are satisfied for some* $\beta \in (0,1)$. *Then there exists a solution* $u \in H^{(2+\beta)/2, 2+\beta}([0,T] \times \mathbb{R}^d)$ *of the Cauchy problem*

$$Lu = 0, \qquad u(0,\cdot) = \psi_0.$$

Proof See Theorem 8.1 in [29], p. 495. □

In the proof of Theorem 6.8 we require the following ramification of this result:

Corollary C.4 *Suppose that*

(C1) $a, b, c : \mathbb{R} \to \mathbb{R}$ *are bounded and Lipschitz continuous,*
(C2) *the function* a *has a bounded Lipschitz continuous derivative and satisfies* $\inf_{y\in\mathbb{R}} a(y) > 0$,
(C3') $\hat{\varepsilon} \in H^{r+2}(\mathbb{R})$ *for some* $r \in (0, 1)$,

and let $f \in C_b^1(\mathbb{R})$. *Then the semilinear PDE*

$$0 = g_t + ag_{yy} + bg_y + cg + f(g), \qquad g(T, \cdot) = \hat{\varepsilon}.$$

has a solution $g \in C_b^{1,2}([0, T] \times \mathbb{R})$.

Proof After setting $a_1(t, x, u, p) \triangleq pa(x)$ and

$$\bar{a}(t, x, u, p) \triangleq -b(x)p - c(x)u - f(u) + pa'(x)$$

we can represent the relevant differential operator as

$$\begin{aligned} Lu &\triangleq u_t - \tfrac{\mathrm{d}}{\mathrm{d}\,x} a_1(t, x, u, u_x) + \bar{a}(t, x, u, u_x) \\ &= u_t - \tfrac{\mathrm{d}}{\mathrm{d}\,x}(u_x a(x)) - b(x)u_x - c(x)u - f(u) + u_x a'(x) \\ &= u_t - a(x)u_{xx} - b(x)u_x - c(x)u - f(u). \end{aligned}$$

Hence if $u \in C_b^{1,2}([0, T]\times\mathbb{R})$ solves $Lu = 0$, $u(0, \cdot) = \hat{\varepsilon}$, then $g(t, x) \triangleq u(T-t, x)$ defines member of $C_b^{1,2}([0, T] \times \mathbb{R})$ that satisfies

$$0 = g_t + ag_{yy} + bg_y + cg + f(g), \quad g(T, \cdot) = \hat{\varepsilon}.$$

We now verify the assumptions of Theorem C.3 for L. Note that

$$a_{11}(t, x, u, p) = \tfrac{\partial a_1(x,t,u,p)}{\partial p_1} = a(x)$$

so (Q1) holds since

$$a_{11}(t, x, u, p)y^2 = a(x)y^2 \geq 0 \quad \text{by (C2)}.$$

Next observe that

$$\begin{aligned} A(t, x, u, p) &= \bar{a}(t, x, u, p) - \tfrac{\partial a_1(t,x,u,p)}{\partial u} p - \tfrac{\partial a_1(t,x,u,p)}{\partial x} \\ &= -b(x)p - c(x)u - f(u) \end{aligned}$$

Thus (Q2) is satisfied since

$$A(t, x, u, 0) = -c(x)u - f(u) \geq -\|c\|_\infty |u| - \|f\|_\infty \geq -b_1 u^2 - b_2$$

with $b_1 \triangleq \|c\|_\infty$ and $b_2 \triangleq \|c\|_\infty + \|f\|_\infty$. To check (Q3) note that by (C1) and (C2) the functions a_1 and $\bar{a}$ are continuous, and a_1 is differentiable; moreover

$$\inf_{x\in\mathbb{R}} a(x)|y|^2 \leq a_{11}(t,x,u,p)y^2 \leq \|\beta\|_\infty |y|^2$$

for all $t \in [0,T]$, and $x,u,p,y \in \mathbb{R}$. In addition for $v = (t,x,u,p) \in [0,T] \times \mathbb{R} \times \mathbb{R} \times \mathbb{R}$

$$\begin{aligned}
&|\bar{a}(v)| + \Big(|a_1(v)| + |\tfrac{\partial a_1(v)}{\partial u}|\Big)(1+|p|) + |a_{11}(v)| \\
&\leq \|b\|_\infty|p| + \|c\|_\infty|u| + \|f\|_\infty + \|a'\|_\infty|p| + |p|\|a\|_\infty(1+|p|) + \|a\|_\infty \\
&\leq (\|a\|_\infty + \|b\|_\infty + \|c\|_\infty + \|f\|_\infty + \|a'\|_\infty)(1+|u|)(1+|p|)^2
\end{aligned}$$

since a, b, c, f, and a' are bounded. Thus (Q3) holds with

$$c_2 \triangleq \|a\|_\infty + \|b\|_\infty + \|c\|_\infty + \|f\|_\infty + \|a'\|_\infty$$

and $c_1 \triangleq \inf_{x\in\mathbb{R}} a(x) > 0$. Finally, by assumptions (C1), (C2) for any compact set $K \subset \mathbb{R}$ the functions

$$\begin{aligned}
&a_1(v) = pa(x), \quad a(v) = -b(x)p - c(x)u - f(u) + pa'(x), \\
&a_{11}(v) = a(x), \quad \tfrac{\partial a_1}{\partial u}(v) = 0, \quad \tfrac{\partial a_1}{\partial p}(v) = a'(x)
\end{aligned}$$

restricted to $[0,T]\times\mathbb{R}\times K\times K$ are Lipschitz continuous in x, u and p, because a, a' b, c and f are bounded and Lipschitz by (C1), (C2) and since $f \in C_b^1(\mathbb{R})$. Hence (Q4)$^{\frac{1}{2}}$ holds as well. Thus by Theorem C.3 the Cauchy problem

$$Lu = 0, \quad u(0,\cdot) = \hat{\varepsilon}$$

has a solution $u \in H^{5/4,5/2}([0,T]\times\mathbb{R}^d) \subset C_b^{1,2}([0,T]\times\mathbb{R}^d)$. Uniqueness follows from standard BSDE arguments; see, e.g., Proposition 4.3 in [20]. □

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